On the rate of approximation in limit theorems for sums of moving averages*

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1. Introduction. Consider a moving average process

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \tag{1}$$

where $(\varepsilon_i, i \in \mathbb{Z})$ are i.i.d. random variables such that X_t is well-defined, that is, the series in (1) converges a.s., or equivalently, in probability. Linear processes as in (1) form a natural class of stationary time series models and include popular parametric classes such as ARMA and ARFIMA. Depending on the rate of decay of the coefficients a_j, the stationary variables X_t in (1) can be weakly or strongly dependent. Usually, weak dependence refers to absolutely convergent coefficients ($\sum_{j=0}^{\infty} |a_j| < \infty$) and strong (or long-range) dependence to the absolutely divergent series ($\sum_{j=0}^{\infty} |a_j| = \infty$). The fundamental Wold decomposition says that every regular stationary Gaussian process can be represented in the form (1), with i.i.d. standard normal innovations $\varepsilon_i, i \in \mathbb{Z}$.

Let

$$S_n := B_n^{-1} \sum_{t=1}^{n} X_t, \qquad S_n(\tau) := B_n^{-1} \sum_{t=1}^{[n\tau]} X_t \quad (\tau \in [0,1]), \tag{2}$$

where B_n is a normalization. Limit behavior of sums S_n and partial sums $S_n(\tau), \tau \in [0,1]$ of linear processes in (1) is well investigated. Several authors (see Davydov [5], Gorodetskii [7], Surgailis [16]) discussed the (functional) convergence of the partial sums process in linear variables X_t with finite variance, to a fractional Brownian motion $B_H(\tau)$ with index $0 < H < 1$. Convergence of partial sums of moving averages of i.i.d. r.v. with infinite variance to an $\alpha-$stable $(0 < \alpha < 2))$ fractional motion was studied in Astrauskas [2], Maejima [11], Avram and Taqqu [3] and other papers. Let us note that convergence of partial sums process $S_n(\tau)$ to a self-similar process (e.g., a fractional Brownian motion) requires a regular growth of normalizing constants B_n, see Lamperti [10]. On the other hand, a central limit theorem for (simple) sums S_n of linear variables as in (1) holds under general assumptions $\mathrm{E}\varepsilon_0^2 < \infty, \mathrm{E}\varepsilon_0 = 0$ and $B_n^2 = \mathrm{E}(\sum_{t=1}^{n} X_t)^2 \to \infty$; see Ibragimov [8], also Ibragimov and Linnik [9], Theorem 18.2.

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Let (ε_i) in (1) belong to the domain of attraction of $\alpha-$stable r.v. η, $0 < \alpha \le 2$. Then it is natural to approximate S_n and $S_n(\tau)$ in (2) by $\alpha-$stable sums

$$Z_n := B_n^{-1} \sum_{t=1}^{n} Y_t, \qquad Z_n(\tau) := B_n^{-1} \sum_{t=1}^{[n\tau]} Y_t \quad (\tau \in [0,1]), \tag{3}$$

respectively, where

$$Y_t := \sum_{j=0}^{\infty} a_j \eta_{t-j}, \tag{4}$$

and $(\eta_i, i \in \mathbb{Z})$ are i.i.d. copies of $\alpha-$stable r.v. η. In many cases, e.g. if η is symmetric or the moving average coefficients (a_j) in (1) are nonnegative, the normalization B_n in (1)-(4) can be chosen so that the distribution of Z_n does not depend on n and coincides with the limit distribution of S_n.

The aim of this note is to obtain a uniform rate of $\alpha-$stable approximation of S_n, namely, the decay rate as $n \to \infty$ of the quantity

$$\Delta_n := \sup_{x \in \mathbb{R}} |\mathrm{P}(S_n \le x) - \mathrm{P}(Z_n \le x)|. \tag{5}$$

Convergence rates in a functional central limit theorem (with the limiting fractional Brownian motion) were obtained in Gorodetskii [7] and Arkashov and Borisov [1] (the later paper contains also some other unpublished references). The reasons for our studying the relative simple "one-dimensional" quantity Δ_n in (5) are the following. Firstly, as mentioned above, a limit distribution of (simple) sums S_n exists under much less restrictive conditions on coefficients (a_j) as compared to partial sums $S_n(\tau)$; moreover, our results apply to situations when $\mathrm{P}(S_n \le x)$ and $\mathrm{P}(Z_n \le x)$ do not converge; see Example 2. Secondly, the results in Gorodetskii [7] and Arkashov and Borisov [1] , obtained under the usual assumption $\mathrm{E}|\varepsilon_0|^3 < \infty$ and some regularity assumptions on (a_j), give a functional convergence rate not better than $n^{-1/8}$, while for the quantity Δ_n one can expect under similar assumptions a much better convergence rate $n^{-1/2}$, similarly as in the i.i.d. case. Indeed, it turns out that in many cases the rate of convergence of Δ_n is the same in the case of i.i.d. summands, namely $\Delta_n = O(n^{-\delta/\alpha})$, assuming the existence of $(2+\delta)-$(pseudo)moment of ε_0, $0 < \delta \le 1$ (see Theorems 1-3 and Assumption $\mathrm{D}(\alpha, \delta)$ below for precise formulations). The above mentioned fact is not surprising since S_n can be represented as a weighted sum in i.i.d. r.v.'s $\varepsilon_i, i \le n$. We also obtain easily verifiable conditions on the coefficients (a_j) in various dependence situations (the so-called cases of short, long or negative memory of (X_t) under which the above mentioned rate of convergence is achieved.

2. Rate of convergence in the central limit theorem for weighted sums of i.i.d. random variables. Let $0 < \alpha \le 2, \alpha \ne 1$, and ε be a r.v. Write $\varepsilon \in D(2) (\alpha = 2)$ if $\mathrm{E}\varepsilon = 0, \sigma^2 := \mathrm{E}\varepsilon^2 < \infty$, and $\varepsilon \in D(\alpha)$ $(0 < \alpha < 2)$ if there exist constants $c_\varepsilon^\pm \ge 0, c_\varepsilon^+ + c_\varepsilon^- > 0$ such that

$$\mathrm{P}(\varepsilon > x) \sim c_\varepsilon^+ x^{-\alpha} \quad (x \to \infty), \qquad \mathrm{P}(\varepsilon < x) \sim c_\varepsilon^- |x|^{-\alpha} \quad (x \to -\infty); \tag{6}$$

moreover, $\mathrm{E}\varepsilon = 0$ for $1 < \alpha < 2$. Condition $\varepsilon \in D(\alpha)$ means that r.v. ε belongs to the domain of normal attraction of $\alpha-$stable distribution (see Ibragimov and Linnik [9]); in other words, if $\varepsilon_1, \varepsilon_2, \dots$ are independent copies of r.v. ε, then

$$n^{-1/\alpha} \sum_{i=1}^{n} \varepsilon_i \to_{\text{law}} \eta, \tag{7}$$

where η is $\alpha-$stable r.v. with the characteristic function

$$\mathrm{E}e^{\mathrm{i}u\eta} = \exp\left\{-|u|^\alpha \omega(u)\right\}.$$

Here

$$\omega(u) \quad := \quad \begin{cases} \sigma^2/2, & \alpha = 2, \\ \frac{\Gamma(2-\alpha)}{1-\alpha}\Big((c_\varepsilon^+ + c_\varepsilon^-)\cos(\pi\alpha/2) - \mathrm{i}(c_\varepsilon^+ - c_\varepsilon^-)\mathrm{sgn}(u)\sin(\pi\alpha/2)\Big), & 0 < \alpha < 2, \alpha \neq 1. \end{cases}$$

We exclude the case $\alpha = 1$ from consideration simply for technical reasons, since in the case $\alpha = 1$ centering and normalization is different from the rest of values of α.

In order to obtain a rate of convergence in the central limit theorem for sums and weighted sums of i.i.d. r.v.'s in $\alpha-$stable domain of attraction, further conditions on the distribution ε must be imposed. Recall that pseudomoment of order $\alpha + \delta > \alpha$ of r.v. $\varepsilon \in \mathrm{D}(\alpha)$ is defined by

$$\kappa_{\alpha,\delta}(\varepsilon) := \int_{\mathbb{R}} |x|^{\alpha+\delta} |\mathrm{d}(\mathrm{P}(\varepsilon \le x) - \mathrm{P}(\eta \le x))|,$$

where η is the $\alpha-$stable r.v. in (7). For $\varepsilon \in D(\alpha), 0 < \alpha \le 2, \alpha \neq 1, 0 < \delta \le 1$, introduce

Assumption $\mathrm{D}(\alpha, \delta)$

(i) If $\alpha = 2$, then $\varepsilon \in \mathrm{D}(2)$ and $\mu_{2+\delta} := \mathrm{E}|\varepsilon|^{2+\delta} < \infty$.

(ii) If $0 < \alpha < 1$, then $\varepsilon \in \mathrm{D}(\alpha)$ and $\kappa_{\alpha,\delta}(\varepsilon) < \infty$. Moreover, if $\alpha + \delta > 1$, then

$$\int_{\mathbb{R}} x \mathrm{d}(\mathrm{P}(\varepsilon \le x) - \mathrm{P}(\eta \le x)) = 0.$$

(iii) If $1 < \alpha < 2$, then $\varepsilon \in \mathrm{D}(\alpha)$, $\mathrm{E}\varepsilon = 0$ and and $\kappa_{\alpha,\delta}(\varepsilon) < \infty$. Moreover, if $\alpha + \delta > 2$, then

$$\int_{\mathbb{R}} x^2 \mathrm{d}(\mathrm{P}(\varepsilon \le x) - \mathrm{P}(\eta \le x)) = 0.$$

Consider weighted sums

$$S_n \; = \; B_n^{-1} \sum_{i \le n} b_{n,i}\varepsilon_i, \qquad Z_n \; = \; B_n^{-1} \sum_{i \le n} b_{n,i}\eta_i, \tag{8}$$

where $\varepsilon_i,\ i \le n$ are i.i.d. copies of ε and $\eta_i,\ i \le n$ are i.i.d. copies of η, coefficients $b_{n,i}, i \le n$ are real and normalizing sequence is defined by

$$B_n \equiv B_n(\alpha) := \Big(\sum_{i \le n} |b_{n,i}|^\alpha\Big)^{1/\alpha}. \tag{9}$$

Here and in what follows $\sum_{i \le n} = \sum_{i=-\infty}^n$. Without loss of generality we assume that $\sigma^2 = 1$ in the case $\alpha = 2$. Note that Z_n has $\alpha-$stable distribution, which does not depend on n (and coincides with the distribution of η) if either η is symmetric, or weights $b_{n,i} \ge 0, i \le n$ are nonnegative and η has arbitrary $\alpha-$stable distribution. Thus, in the case $\alpha = 2$ Z_n for all n is a standard normal random variable. In the general case, the distribution of Z_n depends on n, and Z_n (as well as S_n) does not need to converge to a limit distribution. However, approximation of S_n by Z_n is very natural and the present note discusses the rate of such approximation, namely, the rate of convergence as $n \to \infty$ of the quantity Δ_n from (5) with S_n and Z_n from (8). Introduce the "Lyapunov fraction"

$$L_n(\alpha, \delta) := \Big(\frac{B_n(\alpha+\delta)}{B_n(\alpha)}\Big)^{\alpha+\delta} \; = \; \frac{\sum_{i \le n} |b_{n,i}|^{\alpha+\delta}}{\Big(\sum_{i \le n} |b_{n,i}|^\alpha\Big)^{(\alpha+\delta)/\alpha}}. \tag{10}$$

Theorem 1 *Let* (ε_i) *be i.i.d., whose distribution satisfies Assumption* $\mathrm{D}(\alpha, \delta)$, *for some* $0 < \alpha \leq 2, \alpha \neq 1, 0 < \delta \leq 1$. *Then*

$$\Delta_n \;\leq\; K \begin{cases} \mu_{2+\delta} L_n(2, \delta), & \text{if } \alpha = 2, \\ \max\{\kappa_{\alpha,\delta}, (\kappa_{\alpha,\delta})^{1/(\alpha+\delta+1)}\} L_n(\alpha, \delta), & \text{if } 0 < \alpha < 2, \alpha \neq 1, \end{cases} \tag{11}$$

where K *is a constant depending only on* α, δ.

Remark 1. For $\alpha = 2$, Theorem 1 follows from the classical estimate in Petrov [15], Theorem 5.7, and for $0 < \alpha < 2$, from Paulauskas [14], Corollary 1 (see also Christoph and Wolf [4] and Paulauskas [13]). Although the above mentioned results refer to finite sums of independent r.v.'s only, they can be easily extended to the situation in Theorem 1, by truncating infinite sums in (8) and then letting the level of truncation grow to infinity.

Remark 2. The reason for separating the estimates in the cases $\alpha = 2$ (Gaussian approximation) and $0 < \alpha < 2$ (stable approximation) is that, traditionally, in the former case one uses moments, while in the latter case, moments of the order exceeding α do not exist and one has to use pseudomoments. It is possible to use pseudomoments in the Gaussian case too and to replace the two lines in (11) by a single estimate which holds for any $0 < \alpha \leq 2, \alpha \neq 1$. To do this, instead of the estimate in Petrov [15], one has to use a result due to Nagaev and Rotar' [12] giving a rate of normal approximation in terms of (finite) pseudomoment $\kappa_{2,1}$. The discussion in Paulauskas [13] and [14] shows that extension of these results to the case of finite pseudomoment $\kappa_{2,\delta}$ is not difficult.

3. Rate of stable approximation for sums of moving averages. In this section Theorem 1 is applied to estimate the quantity Δ_n in (5). We compare sums S_n, Z_n of moving averages X_t, Y_t in (1)and (4), respectively, which can be rewritten as the corresponding weighted sums in (8), with weights $b_{n,i}$ given by

$$b_{n,i} \;:=\; \sum_{t=1\vee i}^{n} a_{t-i} \qquad (i \leq n). \tag{12}$$

If (ε_i) satisfy Assumption $\mathrm{D}(\alpha, \delta)$, the series in (1), (4) converge a.s. provided the moving average coefficients satisfy

$$\sum_{j=0}^{\infty} |a_j|^{\alpha} < \infty.$$

Theorem 1 reduces estimation of the uniform distance Δ_n between probability distributions of S_n and the $\alpha-$stable r.v. Z_n in (8), to that of $L_n(\alpha, \delta)$ which depends only on weights $(b_{n,i})$ in (12), or on the moving average coefficients (a_j) in (1). In this section, we study the behavior of $L_n(\alpha, \delta)$ under various assumptions on (a_j) and apply the results to estimation of Δ_n.

Introduce the following notation. Given two sequences $(a_n, n \geq 0)$ and $(b_n, n \geq 0)$, we write $a_n \simeq b_n$ whenever the inequality $C_1 b_n \leq a_n \leq C_2 b_n$ holds for all $n \geq 0$ and some constants $0 < C_1 < C_2 < \infty$, and $a_n \sim b_n$ whenever $\lim_{n\to\infty} a_n/b_n = 1$. Recall that a sequence $(b_j, j \geq 1)$ is regularly varying with index $\gamma \in \mathbb{R}$ if b_j can be represented as $b_j = \Lambda(j) j^{\gamma}$, where $\Lambda(x), x \in [1, \infty)$ is slowly varying at infinity and bounded on bounded intervals. Let (b_j) be regularly varying with index γ, and $a_j \sim b_j$, then (a_j) is also regularly varying with index γ. On the other hand, if (b_j) is regularly varying with index γ and $a_j \simeq b_j$, then (a_j) need not be regularly varying. (Take e.g. $b_j = (1 + j)^{\gamma}$ and $a_j = b_j(2 + (-1)^j)$.)

Consider first the case of positive coefficients (a_j) which are not necessarily regularly varying but are bounded from both sides by positive regularly varying sequences. Note in this case $\mathrm{P}(Z_n \le x) = \mathrm{P}(\eta \le x)$ does not depend on n and coincides with the distribution of η in (7).

Theorem 2 *Let $S_n = B_n^{-1} \sum_{t=1}^n X_t$, where X_t is a moving average process in (1), with i.i.d. innovations (ε_j) satisfying Assumption D(α, δ), $0 < \alpha < 2, \alpha \ne 1, 0 < \delta \le 1$. Let $a_j \simeq (1+j)^{-\beta}$, for some $\beta > 1/\alpha$. Then*

$$\Delta_n \ \le \ C_1 n^{-\delta/\alpha}, \tag{13}$$

where C_1 is some constant depending on α, β, δ as well as on $E|\varepsilon|^{2+\delta}$ $(\alpha = 2)$ or the pseudomoment $\kappa_{\alpha,\delta}$ $(0 < \alpha < 2)$ only. Moreover,

$$B_n(\alpha) \simeq \begin{cases} n^{1-\beta+(1/\alpha)}, & \text{if } 1/\alpha < \beta < 1, \ \alpha > 1, \\ n^{1/\alpha}, & \text{if } \beta > 1, \\ n^{1/\alpha} \log n, & \text{if } \beta = 1, \ \alpha > 1. \end{cases} \tag{14}$$

Proof. According to Theorem 1 and the definitions of $L_n(\alpha, \delta), B_n(\alpha), b_{n,i}$ (see (9), (10), (12)) and assumption $a_j \simeq (1+j)^{-\beta}$, it suffices to consider the case $a_j = (1+j)^{-\beta}$ only. Then (13) follows from (11) and (14), while (14) for $a_j = (1+j)^{-\beta}$ follows by direct calculation (see also the proof of Theorem 3 below). Theorem 2 is proved. □

Next, we discuss the case when the moving average coefficients (a_j) are regularly varying or absolutely summable and may change their sign. The following classification in terms of memory is often used to characterize the dependence structure and the limit behavior of various functionals of X_t (see, e.g., [6]). Correspondingly, we shall distinguish between the following cases, or assumptions

(I) $\sum_{j=0}^\infty |a_j| < \infty, \ \sum_{j=0}^\infty a_j \ne 0$;

(II) $a_j = \Lambda(j) j^{-\beta}, \ 1/\alpha < \beta < 1, \ 1 < \alpha \le 2$, Λ is slowly varying at infinity;

(III) $a_j = \Lambda(j) j^{-\beta}, \ \sum_{j=0}^\infty a_j = 0, \ \beta > 1$, Λ is slowly varying at infinity.

Cases (I), (II), (III) are usually called short memory, long memory and negative memory, respectively. The following theorem shows that in cases (I) and (II), the rate of $\alpha-$stable approximation is the same as in the i.i.d. case.

Theorem 3 *Let X_t be a moving average as in (1), with i.i.d. innovations (ε_j) satisfying Assumption D(α, δ), $0 < \alpha \le 2, \alpha \ne 1, 0 < \delta \le 1$. Assume that the moving average coefficients (a_j) satisfy one of assumptions (I) - (III) above. Then:*

(i) Under assumptions (I) or and (II) (short or long memory),

$$\Delta_n \ \le \ C_2 n^{-\delta/\alpha}. \tag{15}$$

(ii) Under assumption (III) (negative memory)

$$\Delta_n \ \le \ C_2 \begin{cases} n^{-\delta/2}, & \text{if } 1 \vee (1/\alpha) < \beta < 1 + (1/(\alpha+\delta)), \\ (\Lambda(n) n^{1+1/\alpha-\beta})^{-(\alpha+\delta)} = O(B_n(\alpha)^{-(\alpha+\delta)}), & \text{if } 1 + (1/(\alpha+\delta)) < \beta < 1 + (1/\alpha). \end{cases} \tag{16}$$

In (15) and (16), C_2 is a constant depending on the sequence (a_j) as well as on the $(2+\delta)-$moment $(\alpha = 2)$ or the pseudomoment $\kappa_{\alpha,\delta}$ $(0 < \alpha < 2)$ only, otherwise independent of the distribution of ε.

Theorem 3 follows easily from Theorem 1 and Lemma 1 below, which gives asymptotical behavior of $B_n(\alpha)$ under conditions (I) - (III).

REMARK 3. It is not difficult to explicitly give the dependence of constants C_1, C_2 in Theorems 2 and 3 on moments and pseudomoments. Namely, similarly to (11), one can show that there are some constants $\widetilde{C}_i, i = 1, 2$ depending on α, δ and the sequence (a_j) only, such that $C_i = \mu_{2+\delta}\widetilde{C}_i$ if $\alpha = 2$, and $C_i = \max\{\kappa_{\alpha,\delta}, (\kappa_{\alpha,\delta})^{1/(\alpha+\delta+1)}\}\widetilde{C}_i$ if $0 < \alpha < 2$.

REMARK 4. An open question is whether the rate of convergence in case (III), $1 \vee (1/\alpha) < \beta < 1 + (1/(\alpha+\delta))$ given in (16), is optimal. The fact that $B_n(\alpha)$ is bounded for $\beta > 1 + (1/(\alpha+\delta))$ is in favor of a positive answer to this question.

In Lemma 1 below, $\Lambda(x), x \ge 0$ is a function slowly varying at infinity, which we assume to be strictly positive for x large enough.

Lemma 1

(i) Let $\sum_{j=0}^\infty |a_j|^{1\wedge\alpha} < \infty$, $\sum_{j=0}^\infty a_j \ne 0, \alpha > 0$. Then

$$B_n^\alpha(\alpha) \sim |\phi_\infty|^\alpha n, \tag{17}$$

where $\phi_\infty := \sum_{i=0}^\infty a_i$.

(ii) Let $a_j = \Lambda(j)j^{-\beta}$, where $1/\alpha < \beta < 1, \alpha > 1$. Then

$$B_n^\alpha(\alpha) \sim \ c(\alpha,\beta)\Lambda^\alpha(n)n^{\alpha(1-\beta)+1}. \tag{18}$$

where the constant $c(\alpha,\beta)$ is given in (19) below.

(iii) Let $a_j = \Lambda(j)j^{-\beta}$, where $1 \vee (1/\alpha) < \beta < 1 + (1/\alpha), \alpha > 0$; $\sum_{j=0}^\infty a_j = 0$. Then relation (18) *holds, with $c(\alpha,\beta)$ is given in (19).*

(iv) Let $a_j = \Lambda(j)j^{-\beta}$, where $\beta > 1+(1/\alpha), \alpha > 0$; $\sum_{j=0}^\infty a_j = 0$. Then $B_n^\alpha(\alpha) \to 2\sum_{i=0}^\infty \left|\sum_{j=i+1}^\infty a_j\right|^\alpha >$ 0.

Proof. (i) Note that $\sum_{j=0}^\infty |a_j|^{1\wedge\alpha} < \infty$ implies $\sum_{j=0}^\infty |a_j| < \infty$, for any $\alpha > 0$. Let $\phi_n := \sum_{i=0}^n a_i$, $\Phi_n := \sum_{i\ge n} |a_i|$, $n \ge 0$. Then $|\phi_{n+i} - \phi_i| \le \Phi_{i+1}$ $(i, n \ge 0)$, $\phi_n \to \phi_\infty, \Phi_n \to 0$ $(n \to \infty)$. Write

$$n^{-1}B_n^\alpha(\alpha) = D_n' + D_n'', \quad D_n' := n^{-1}\sum_{i=0}^{n-1}|\phi_i|^\alpha, \quad D_n'' := n^{-1}\sum_{i=0}^{\infty}|\phi_{n+i} - \phi_i|^\alpha.$$

By the dominated convergence theorem, $\lim_{n\to\infty} D_n' = |\phi_\infty|^\alpha$. It remains to show $D_n'' = o(1)$. Assume first $\alpha \ge 1$. Then $|\phi_{n+i} - \phi_i|^\alpha \le \Phi_0^{\alpha-1}|\phi_{n+i} - \phi_i|$ and therefore

$$D_n'' \le \Phi_0^{\alpha-1}n^{-1}\sum_{i=0}^\infty(|a_{i+1}| + \ldots + |a_{n+i}|) \le \Phi_0^{\alpha-1}n^{-1}(\Phi_0 + \ldots + \Phi_n) \to 0 \quad (n \to \infty).$$

Next, let $0 < \alpha < 1$. Then $\sum_{j=0}^\infty |a_j|^\alpha < \infty$ and we similarly obtain

$$D_n'' \le n^{-1}\sum_{i=0}^\infty(|a_{i+1}|^\alpha + \ldots + |a_{n+i}|^\alpha) \le n^{-1}\sum_{i=1}^n\sum_{j=i}^\infty |a_j|^\alpha \to 0 \quad (n \to \infty),$$

proving (17).

(ii) Let $\phi_n = \sum_{i=0}^n a_i$ as in part (i). Observe, by the dominated convergence theorem,

$$\gamma_n := \frac{c_n}{\Lambda(n)n^{1-\beta}} \;=\; \sum_{i=0}^n \frac{\Lambda(i)}{\Lambda(n)}\Big(\frac{i}{n}\Big)^{-\beta}\frac{1}{n} \;\to\; \int_0^1 x^{-\beta}\mathrm{d}x \;=\; (1-\beta)^{-1}.$$

Then $B_n^\alpha(\alpha)/\Lambda^\alpha(n)n^{\alpha(1-\beta)+1} = D_n + D'_{n,K} + D''_{n,K}$, where

$$\begin{aligned} D_n \;&:=\; \sum_{i=0}^n |\gamma_i|^\alpha \frac{\Lambda^\alpha(i)}{\Lambda^\alpha(n)}\Big(\frac{i}{n}\Big)^{\alpha(1-\beta)}\frac{1}{n}, \\ D'_n(K) \;&:=\; \sum_{i=0}^{Kn} \Big|\gamma_{n+i}\frac{\Lambda(n+i)}{\Lambda(n)}\Big(\frac{n+i}{n}\Big)^{1-\beta} - \gamma_i \frac{\Lambda(i)}{\Lambda(n)}\Big(\frac{i}{n}\Big)^{1-\beta}\Big|^\alpha \frac{1}{n}, \\ D''_n(K) \;&:=\; \Lambda^{-\alpha}(n)n^{-\alpha(1-\beta)-1}\sum_{i>Kn}|\phi_{n+i}-\phi_i|^\alpha, \end{aligned}$$

where K is a large number. By the dominated convergence theorem, for any $K<\infty$,

$$\begin{aligned} D_n \;&\to\; (1-\beta)^{-\alpha}\int_0^1 x^{\alpha(1-\beta)}\mathrm{d}x \;=:\, D, \\ D'_n(K) \;&\to\; (1-\beta)^{-\alpha}\int_0^K ((1+x)^{1-\beta}-x^{1-\beta})^\alpha \mathrm{d}x \;=:\, D'(K). \end{aligned}$$

Rewrite

$$D''_n(K) = n^{-\alpha}\sum_{i>nK}\Big|\sum_{j=1}^n \frac{\Lambda(i+j)}{\Lambda(n)}\Big(\frac{i+j}{n}\Big)^{-\beta}\Big|^\alpha \frac{1}{n}.$$

We shall use the well-known property of slowly varying functions: for any $\theta>0$,

$$\sup_{x\in[1,\infty)}\frac{1}{x^\theta}\Big|\frac{\Lambda(nx)}{\Lambda(n)}-1\Big| \;\to\; 0 \qquad (n\to\infty).$$

Therefore, for any (sufficiently small) $\theta>0$ one can find a constant $C=C_\theta$ such that

$$\begin{aligned} D''_n(K) \;&\le\; Cn^{-\alpha}\sum_{i>nK}\Big(\sum_{j=1}^n\Big(\frac{i+j}{n}\Big)^{\theta-\beta}\Big)^\alpha\frac{1}{n} \;\le\; C\int_K^\infty\Big(\int_0^1 (x+y)^{\theta-\beta}\mathrm{d}y\Big)^\alpha \mathrm{d}x \\ &\le\; C\int_K^\infty x^{\alpha(\theta-\beta)}\mathrm{d}x \;\le\; CK^{1+\alpha(\theta-\beta)} \;=\; o(1), \quad (K\to\infty), \end{aligned}$$

provided $\theta>0$ was chosen so that $\theta<\beta-(1/\alpha)$. Furthermore, $D'(K)\to D' := (1-\beta)^{-\alpha}\int_0^\infty((1+x)^{1-\beta}-x^{1-\beta})^\alpha\mathrm{d}x$ as $K\to\infty$. We have proved that $\lim_{n\to\infty}B_n^\alpha(\alpha)/\Lambda^\alpha(n)n^{\alpha(1-\beta)+1} = D+D' =: c(\alpha,\beta)$, where

$$c(\alpha,\beta) \;=\; |1-\beta|^{-\alpha}\left(\int_0^1 x^{\alpha(1-\beta)}\mathrm{d}x + \int_0^\infty \left|(1+x)^{1-\beta}-x^{1-\beta}\right|^\alpha \mathrm{d}x\right). \tag{19}$$

In particular,

$$c(2,\beta) = \frac{\Gamma^2(2-\beta)}{(1-\beta)^2\Gamma(4-2\beta)|\cos(\pi\beta)|}$$

see Taqqu [17], (9.3).

(iii). Observe $\phi_n = -\sum_{i=n+1}^{\infty} a_i$ and

$$\gamma_n := \frac{\phi_n}{\Lambda(n) n^{1-\beta}} \;=\; -\sum_{i=n+1}^{\infty} \frac{\Lambda(i)}{\Lambda(n)} \Big(\frac{i}{n}\Big)^{-\beta} \frac{1}{n} \;\to\; -\int_1^{\infty} x^{-\beta} \mathrm{d}x \;=\; -(\beta-1)^{-1}.$$

The rest of the proof of (18) is completely analogous to the case (ii).

(iv). Follows by $\sum_{i=0}^{\infty} \big| \sum_{j=i+1}^{\infty} a_j \big|^{\alpha} < \infty$. Lemma 1 is proved. □

We end the paper with two examples where the classical approximation rate $n^{-\delta/\alpha}$ for $S_n = B_n^{-1} \sum_{t=1}^{n} X_t$ by $\alpha-$stable $Z_n = B_n^{-1} \sum_{t=1}^{n} Y_t$ takes place. In the first example, the distribution of Z_n does not depend on n and coincides with the limit distribution for S_n but the normalizing sequence B_n is not regularly varying, so that partial sums process of X_t does not converges (see Introduction). In the second example, the approximating $\alpha-$stable distribution Z_n does not converge as its skewness parameter oscillates with n.

Example 1 Let $n(k) \ge 1$ $(k = 1, 2, \ldots)$ be a an increasing sequence of integers such that $\lim_{k\to\infty} n(k+1)/n(k) = \infty$.

Fix $\beta \in (1/\alpha, 1), 1 < \alpha \le 2$. Set

$$a_j \;:=\; (1+j)^{-\beta} \begin{cases} 2, & \text{if } n(k) \le j < n(k+1),\ k \text{ odd}, \\ 1, & \text{if } n(k) \le j < n(k+1),\ k \text{ even}. \end{cases}$$

Clearly, $a_j \simeq (1+j)^{-\beta}$ and therefore $B_n = B_n(\alpha) \simeq n^{1-\beta+(1/\alpha)}$, see Theorem 2. To show that (B_n) is not regulary varying, it suffices to show that (B_n^{α}) is not regularily varying with index $H := \alpha(1-\beta)+1$. The last fact follows from

$$\liminf_{k\to\infty} \frac{B^{\alpha}_{2n(2k)}}{B^{\alpha}_{n(2k)}} \;<\; 2^H. \tag{20}$$

Similarly as in the proof of Lemma 1 (ii),

$$B_n^{\alpha} \;\sim\; \int_0^n \Big(\int_0^x u^{-\beta} h(u) \mathrm{d}u\Big)^{\alpha} \mathrm{d}x + \int_0^{\infty} \Big(\int_x^{n+x} u^{-\beta} h(u) \mathrm{d}u\Big)^{\alpha} \mathrm{d}x \;=:\; D'_n + D''_n,$$

where $h(u) := 2$ if $n(k) \le u < n(k+1)$ and k is odd, $h(u) := 1$ if $n(k) \le u < n(k+1)$ and k is even. Note $n(2k)$ is even and therefore

$$h(u) = \begin{cases} 2, & n(2k-1) \le u < n(2k), \\ 1, & n(2k) \le u < n(2k+1). \end{cases}$$

Using $n(2k-1)/n(2k) \to 0, n(2k+1)/n(2k) \to \infty$ and definition of $h(u)$, it is easy to show that

$$D'_{n(2k)} \;\sim\; 2^{\alpha} \int_0^{n(2k)} \Big(\int_0^x u^{-\beta} \mathrm{d}u\Big)^{\alpha} \mathrm{d}x \;=\; 2^{\alpha} (n(2k))^H c_1, \tag{21}$$

$$D''_{n(2k)} \;\sim\; \int_0^{\infty} \Big(\int_x^{n(2k)+x} u^{-\beta} \mathrm{d}u\Big)^{\alpha} \mathrm{d}x \;=\; (n(2k))^H c_2, \tag{22}$$

where

$$c_1 := \int_0^1 \Big(\int_0^x u^{-\beta} \mathrm{d}u\Big)^{\alpha} \mathrm{d}x, \qquad c_2 := \int_0^{\infty} \Big(\int_x^{1+x} u^{-\beta} \mathrm{d}u\Big)^{\alpha} \mathrm{d}x.$$

In a similar way,

$$D''_{2n(2k)} \sim \int_0^\infty \Big(\int_x^{2n(2k)+x} u^{-\beta}\mathrm{d}u\Big)^2 \mathrm{d}x = 2^{\alpha(1-\beta)+1}(n(2k))^H c_2, \tag{23}$$

while

$$\begin{aligned} D'_{2n(2k)} &\sim D'_{n(2k)} + \int_{n(2k)}^{2n(2k)} \Big(\int_0^x u^{-\beta}h(u)\mathrm{d}u\Big)^\alpha \mathrm{d}x \\ &\sim D'_{n(2k)} + \int_{n(2k)}^{2n(2k)} \Big(2\int_0^{n(2k)} u^{-\beta}\mathrm{d}u + \int_{n(2k)}^x u^{-\beta}\mathrm{d}u\Big)^\alpha \mathrm{d}x \\ &= D'_{n(2k)} + (n(2k))^H c_3, \end{aligned} \tag{24}$$

where

$$c_3 := (1-\beta)^{-\alpha}\int_1^2 (1+x^{1-\beta})^\alpha \mathrm{d}x.$$

From (21)-(24) we obtain

$$\frac{B^\alpha_{2n(2k)}}{B^\alpha_{n(2k)}} \sim \frac{D'_{2n(2k)} + D''_{2n(2k)}}{D'_{n(2k)} + D''_{n(2k)}} \sim \frac{2^\alpha c_1 + c_3 + 2^{\alpha(1-\beta)+1}c_2}{2^\alpha c_1 + c_2}.$$

The desired inequality (20) now follows from $c_3 < 2^\alpha c_1(2^{\alpha(1-\beta)+1}-1)$, or

$$\int_1^2 (1+x^{1-\beta})^\alpha \mathrm{d}x < \frac{2^\alpha(2^{\alpha(1-\beta)+1}-1)}{1+\alpha(1-\beta)},$$

The last inequality holds because the r.h.s. equals $\int_1^2(2x^{1-\beta})^\alpha\mathrm{d}x$. This proves (20).

EXAMPLE 2 Let $n(k) \ge 1, n(k+1)/n(k) \to \infty$ be the same as in Example 1. For $\beta \in (1/\alpha, 1), 1 < \alpha < 2$, let

$$a_j := (-1)^k(1+j)^{-\beta}, \quad n(k) \le j < n(k+1), \quad k \ge 1.$$

The above sequence (a_j) is not regularly varying and does not satisfy Theorem 2 nor Theorem 3. However, Theorem 1 is applicable since

$$B^\alpha_n = \sum_{i\le n} |b_{n,i}|^\alpha \simeq n^{\alpha(1-\beta)+1}, \tag{25}$$

see below, and therefore $\Delta_n = \sup_n |\mathrm{P}(S_n \le x) - \mathrm{P}(Z_n \le x)| \le Cn^{-\delta/\alpha}$, provided the i.i.d. r.v.'s (ε_i) satisfy assumption $\mathrm{D}(\alpha,\delta)$. Let $c^+_\varepsilon \ne c^-_\varepsilon$ in (6) so that η in (7) is not symmetric and therefore $Z_n = \sum_{i\le n} b_{n,i}\eta_i$ is not symmetric.

The skewness parameter of Z_n equals $\beta_\eta Q_n$, where $\beta_\eta := (c^+_\varepsilon - c^-_\varepsilon)/(c^+_\varepsilon + c^-_\varepsilon)$ is the skewness parameter of η in (7) and

$$Q_n := B_n^{-\alpha}\sum_{i\le n}|b_{n,i}|^\alpha \mathrm{sgn}(b_{n,i}).$$

Clearly, if $\beta_\eta \ne 0$ and the sequence (Q_n) does not converge, i.e.

$$\limsup Q_n > \liminf Q_n, \tag{26}$$

then (Z_n) does not converge, too (in distribution). Let us prove (25). The relation $B_n^\alpha = O(n^{\alpha(1-\beta)+1})$ follows immediately from $|a_j| \le (1+j)^{-\beta}$, therefore it suffices to show the lower bound

$$B_n^\alpha > c_1 n^{\alpha(1-\beta)+1}, \tag{27}$$

for some constant $c_1 > 0$. Let $\phi_n := \sum_{i=0}^n a_i$. We claim that there are constants $c_2, K > 0$ such that for any k

$$|\phi_i| > c_2 i^{1-\beta}, \qquad \mathrm{sgn}(\phi_i) = (-1)^k \qquad (Kn(k) < i \le n(k+1)). \tag{28}$$

For $n(k) < n \le n(k+1)$, by definition of a_j, we have

$$\phi_n - \phi_{n(k)} = \sum_{j=n(k)+1}^{n} a_j = (-1)^k \sum_{j=n(k)+1}^{n} (1+j)^{-\beta}.$$

Hence, there is a constant $c_5 > 0$ such that

$$|\phi_i - \phi_{n(k)}| > c_5 i^{1-\beta}, \qquad \mathrm{sgn}(\phi_i - \phi_{n(k)}) = (-1)^k$$

holds for all $2n(k) < i \le n(k+1)$. Take $c_2 = c_5/2$, then

$$|\phi_i| \ge |\phi_i - \phi_{n(k)}| - |\phi_{n(k)}| > c_5 i^{1-\beta} - (1-\beta)^{-1}(n(k))^{1-\beta} > c_2 i^{1-\beta}, \qquad \mathrm{sgn}(\phi_i) = (-1)^k$$

holds for all $Kn(k) < i \le n(k+1)$, provided K is chosen large enough. This proves (28).

Now (27) follows from (28). Indeed, if $n > 2Kn(k)$, then

$$B_n^\alpha \;\ge\; \sum_{Kn(k)<i\le n} |\phi_i|^\alpha \;>\; c_2^\alpha \sum_{Kn(k)}^{n} i^{\alpha(1-\beta)} \;>\; c_3 n^{\alpha(1-\beta)+1},$$

for some constant $c_3 > 0$ independent of n. Similarly, if $n \le 2Kn(k)$, or $n(k) \ge n/(2K)$, then

$$B_n^\alpha > \; c_1^\alpha \sum_{2Kn(k-1)<i\le n(k)} i^{\alpha(1-\beta)} \;>\; c_4(n(k))^{\alpha(1-\beta)+1} \;>\; c_4 n^{\alpha(1-\beta)+1}$$

for some constant $c_4 > 0$ independent of n. Thus, we proved (27) and (25).

Let us check (26). It suffices to take the lim sup and lim inf along a subsequence $\bar n(k) \to \infty$, $\bar n(k) := (n(k) + n(k+1))/2$. The integer $\bar n(k)$ is in middle of the interval $(n(k), n(k+1))$ and $\bar n(k)/n(k) \to \infty$ according to our construction of $(n(k))$. We have $Q_n = Q_{n,1} + Q_{n,2}$, where $Q_{n,1} := B_n^{-\alpha}\sum_{i=0}^n |b_{n,i}|^\alpha \mathrm{sgn}(b_{n,i})$, $\quad Q_{n,2} := B_n^{-\alpha}\sum_{i<0} |b_{n,i}|^\alpha \mathrm{sgn}(b_{n,i})$. Note $Q_{n,1} \;=\; B_n^{-\alpha}\sum_{i=0}^n |\phi_i|^\alpha \mathrm{sgn}(\phi_i)$. Using (28),

$$Q_{n,1} \;=\; B_n^{-\alpha}\sum_{i=0}^{Kn(k)} |\phi_i|^\alpha \mathrm{sgn}(\phi_i) \;+\; (-1)^k B_n^{-\alpha} \sum_{i=Kn(k)+1}^{n} |\phi_i|^\alpha \;=:\; Q_{n,3} + (-1)^k Q_{n,4}$$

Then (26) follows from

$$\liminf_{k\to\infty} Q_{\bar n(k),4} > 0, \qquad \limsup_{k\to\infty} |Q_{\bar n(k),i}| = 0 \qquad (i = 2,3). \tag{29}$$

The proof of (29) follows from $B_n^\alpha \simeq n^{\alpha(1-\beta)+1}$, see (25), and a similar argument as in Example 1.

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