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A NOTE ON SELF-NORMALIZATION FOR A SIMPLE SPATIAL AUTOREGRESSIVE MODEL*

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Abstract. In this paper, we consider the problem of self-normalization for one rather simple autoregressive model $X_{t,s} = aX_{t-1,s} + bX_{t,s-1} + \varepsilon_{t,s}$ on a two-dimensional lattice. We show that there is some similarity between this problem and the corresponding problem for $AR(1)$ time series model.

Keywords: autoregressive models, self-normalization, random fields.

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1. INTRODUCTION AND FORMULATION OF RESULTS

Nearly a hundred years passed from the famous paper by Gosset [5] written under the pseudonym "Student," and now self-normalization is widely used in probability and mathematical statistics. Let $X_i,\ i \geqslant 1$, be a stationary mean-zero sequence. Then one considers the sum $S_n = \sum_{i=1}^n X_i$ normalized by the square root of the sum of squares $V_n^2 = \sum_{i=1}^n X_i^2$. In the case of independent and identically distributed (i.i.d.) random variables, it is justified by the fact that usually good normalization is achieved by the square root of variance of S_n which is nEX_1^2. Due to the Law of Large Numbers, V_n^2 is a good approximation for the last quantity. At present, limit behavior (limit theorems with rates of convergence and asymptotic expansions, large deviations) of self-normalized sequence $V_n^{-1}S_n$ in the case of i.i.d. is deeply investigated. There is a large amount of literature, and we refer to [3],[4], [9], [10]. The situation becomes more complicated for sequences of dependent random variables, and solution of the problem for general stationary sequences is far from being completed. Some new effects comparing with i.i.d. case were noticed in [7] for exchangeable random variables. Recently, in [8], a specific form of dependence was considered, precisely, it was supposed that $X_i,\ i \in \mathbf{Z}$, is $AR(1)$ process obtained as a solution of the equation

$$X_i = \rho X_{i-1} + \varepsilon_i,$$

where $|\rho| < 1$ and $\varepsilon_i,\ i \in \mathbf{Z}$, is a sequence of i.i.d. random variables with $E\varepsilon_1 = 0$ and finite variance. It is well known that, in this case, $X_i,\ i \in \mathbf{Z}$, is a stationary sequence

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which can be expressed as the infinite series

$$X_i = \sum_{j=1}^{\infty} \rho^j \varepsilon_{i-j}. \tag{1}$$

In [8], two results are proved. Let us denote by $N(a, \sigma^2)$ the normal distribution with mean a and variance σ^2.

THEOREM A [Theorem 1 of [8]]. *If X_i, $i \in \mathbf{Z}$, is defined by* (1) *with $E\varepsilon_1 = 0$ and $E\varepsilon_1^2 < \infty$, then*

$$V_n^{-1} S_n \xrightarrow{\mathcal{D}} N\Big(0, \frac{1+\rho}{1-\rho}\Big).$$

In practice, having only observed values X_i, $1 \leqslant i \leqslant n$, the appearance of the unknown quantity ρ as a parameter of the limit distribution is unpleasant. One way to overcome this difficulty would be taking some estimator $\hat{\rho}_n$ (for example, the least-square estimator) and proving that

$$\Big(\frac{1-\hat{\rho}_n}{1+\hat{\rho}_n}\Big)^{1/2} V_n^{-1} S_n$$

is asymptotically standard normal. However, the authors of the above-mentioned paper suggested another (to our mind, simpler) approach by taking self-normalization by blocks. Assuming that $n = mN$, with both m, N integers, they introduced the random variables

$$Y_j = \sum_{(j-1)m < i \leqslant jm} X_i, \quad j = 1, 2, \ldots, N,$$

and

$$U_n^2 = Y_1^2 + \ldots + Y_N^2.$$

THEOREM B [Theorem 2 of [8]]. *Under the conditions of Theorem* A*,*

$$U_n^{-1} S_n \xrightarrow{\mathcal{D}} N(0, 1),$$

provided that $m \to \infty$ and $m/n \to 0$ as $n \to \infty$.

Juodis and Račkauskas [8] proved some results on self-normalization for more general linear processes, but we confine ourselves with these two results, since our goal is to consider self-normalization for sums of multi-indexed random variables (random fields). A general problem can be formulated as follows. Let $X_{\bar{k}}$, $\bar{k} \in \mathbf{Z}^d$, be a stationary random field, and let D_n be some sequence of increasing ($D_n \subset D_{n+1}$) finite subsets (with growing cardinality) of $\mathbf{Z}^d$. Then one is interested in the limit behavior of the self-normalized sequence

$$\frac{\sum_{D_n} X_{\bar{k}}}{(\sum_{D_n} X_{\bar{k}}^2)^{1/2}}.$$

If $X_{\bar{k}},\ \bar{k} \in \mathbf{Z}^d$, are i.i.d. random variables, then the problem can be easily transformed to the self-normalization problem for usual sums S_n only over some subsequences (depending on the cardinalities of sets D_n). But if the random field has some dependence structure, then the problem is even more difficult comparing with processes, since this dependence structure (spatial dependence) can be more complicated comparing with dependence in time series. Another degree of complication is added by sets D_n, and there can be even interplay between the dependence structure and the form of sets of summation.

The goal of this paper is rather modest: to generalize above formulated two results for the simple spatial autoregression model and very simple sets D_n in the case $d = 2$. We consider one of the simplest autoregression models,

$$X_{t,s} = aX_{t-1,s} + bX_{t,s-1} + \varepsilon_{t,s}. \tag{2}$$

We suppose that $\varepsilon_{t,s}$, $(t, s) \in \mathbf{Z}^2$ are i.i.d. random variables with $E\varepsilon_{t,s} = 0$ and $E\varepsilon_{t,s}^2 = 1$. We also assume that $|a| + |b| < 1$; this condition guarantees that there is a stationary solution for (2) which has the expression

$$X_{t,s} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \varepsilon_{t-j,s-k+j}. \tag{3}$$

For integers $l_1,\ l_2$, let us denote $\gamma_{(l_1,l_2)} = EX_{t,s}X_{t+l_1,s+l_2}$, where $X_{t,s}$ is from (2). In [2] (see also [1]), it is shown that

$$\gamma_{(0,0)} := ((1+a+b))(1+a-b)(1-a+b)(1-a-b))^{-1/2},$$
$$\gamma_{(-1,1)} := \begin{cases} ((1-a^2-b^2)\gamma_{(0,0)} - 1)(2ab\gamma_{(0,0)})^{-1} & \text{if} \quad ab \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\sigma^2(a, b) = (1-a^2-b^2)(1-a-b)^{-2}(1+2ab\gamma_{(-1,1)})^{-1}.$$

Finally, let us denote $\bar{M}_n = (M_{n,1},\ M_{n,2})$ and

$$D_n = \{(t, s)\colon\ (t, s) \in \mathbf{Z}^2, \quad 1 \leqslant t \leqslant M_{n,1},\ 1 \leqslant s \leqslant M_{n,2}\}.$$

(In the sequel, we suppress the index n in the notation when there is no danger of confusion.) Now our first result can be formulated as follows.

THEOREM 1. *If $X_{t,s}$ is a stationary solution of* (2), *random variables $\varepsilon_{t,s}$ satisfy the above formulated conditions, and* $\min\{M_1, M_2\} \to \infty$ *as $n \to \infty$, then*

$$\frac{\sum_{(t,s)\in D_n} X_{t,s}}{(\sum_{(t,s)\in D_n} X_{t,s}^2)^{1/2}} \xrightarrow{\mathcal{D}} N(0, \sigma^2(a,b)). \tag{4}$$

Such a result probably holds for more general spatial autoregression processes

$$X_{\bar{t}} = \sum_{\bar{k}\in\Lambda} a_{\bar{k}} X_{\bar{t}-\bar{k}} + \epsilon_{\bar{t}} \tag{5}$$

or general linear fields

$$X_{\bar{t}} = \sum_{\bar{k}\in\mathbf{Z}^d} b_{\bar{k}} \varepsilon_{\bar{t}-\bar{k}}$$

under some mild conditions on the coefficients $a_{\bar{k}}$ and $b_{\bar{k}}$. Here $\bar{t} = (t_1, \dots, t_d)$, $\bar{b} = (b_1, \dots, b_d)$, $\bar{a} = (a_1, \dots, a_d)$, Λ is a subset of $\mathbf{Z}^d \setminus 0$, $\epsilon_{\bar{t}}, \bar{t} \in \mathbf{Z}^d$, are i.i.d. random variables with mean zero and finite variance. Taking the simple set

$$\Lambda_1 = \Big\{ i \in \mathbf{Z}^d\colon 0 < \sum_{j=1}^{d} i_j \leqslant 1,\ 0 \leqslant i_j,\ j = 1, \dots, d \Big\}$$

in (5), we get the following generalization of Theorem 1 which we formulate without proof, since it is completely similar to that of Theorem 1, only with a more complicated expression of variance of limit law.

Denote

$$H_n = \{(t_1, \dots, t_d) \in \mathbf{Z}^d, 1 \leqslant t_i \leqslant M_{n,i}\}.$$

THEOREM 1A. *Let a stationary real-valued random field $X_{\bar{t}}$, $\bar{t} \in \mathbf{Z}^d$, be defined by* (5) *with the set Λ_1 and $\sum_{\bar{k}\in\Lambda_1} |a_{\bar{k}}| < 1$. Suppose that $E\varepsilon_{\bar{t}} = 0$, $E\varepsilon_{\bar{t}}^2 < \infty$, and* $\min_i M_{n,i} \to \infty$ *as $n \to \infty$. Then*

$$\frac{\sum_{\bar{t}\in H_n} X_{\bar{t}}}{\big(\sum_{\bar{t}\in H_n} X_{\bar{t}}^2\big)^{1/2}} \xrightarrow{\mathcal{D}} N(0, \sigma_{\tilde{a}}^2).$$

Here $\sigma_{\tilde{a}}^2$ is a constant depending on $\tilde{a} = \{a_{\bar{k}},\ \bar{k} \in \Lambda_1\}$ and dimension d.

This result shows that the dimension d of the indices is unimportant in the problem of self-normalization (on the contrary, as is shown in [11], the dimension is very important considering the growth of variance of a spatial autoregression model with unit root).

The dependence of the variance of the limit normal law in (4) is more complicated as compared with corresponding result in Theorem A. Therefore, the attempt to get rid of this dependence by estimating unknown parameters a and b does not look promising, while the approach proposed in Theorem B is attractive. To this aim we divide the set D_n into smaller rectangles. Assuming that $1 \leqslant m_1 \leqslant M_1$ and $1 \leqslant m_2 \leqslant M_2$ are integers (also depending on n) and such that $I := M_1 m_1^{-1}$ and $J := M_2 m_2^{-1}$ are integers, we set $m_{1,i} := m_1(i-1),\ m_{2,j} := m_2(j-1)$, and

$$D_{i,j} = \left\{(t,s)\colon\ m_{1,i}+1 \leqslant t \leqslant m_{1,i+1},\ m_{2,j}+1 \leqslant s \leqslant m_{2,j+1}\right\},\\ 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant J.$$

If we define

$$Y_{i,j} = \sum_{(t,s)\in D_{i,j}} X_{t,s},$$

then, since $D_n = \cup_{i=1}^{I} \cup_{j=1}^{J} D_{i,j}$, we clearly have

$$\sum_{(t,s)\in D_n} X_{t,s} = \sum_{i=1}^{I}\sum_{j=1}^{J} Y_{i,j}.$$

Since the two sums from the last equality will be used several times, we introduce the notation

$$\sum\nolimits^{*} := \sum_{(t,s)\in D_n}, \quad \sum\nolimits_{*} := \sum_{i=1}^{I}\sum_{j=1}^{J}.$$

THEOREM 2. *If conditions of Theorem* 1 *are satisfied and additionally*

$$\min(m_1,\ m_2) \longrightarrow \infty \quad \textit{and} \quad \frac{m_1 m_2}{M_1 M_2} \longrightarrow 0$$

as $n \to \infty$, *then*

$$\frac{\sum^{*} X_{t,s}}{(\sum_{*} Y_{i,j}^2)^{1/2}} \xrightarrow{\mathcal{D}} N(0,1).$$

2. AUXILIARY RESULTS

In this section, we collected some supplementary results which are used in the proof of Theorems 1 and 2. To formulate them we need some additional notation. The inner radius $d(G)$ of a set $G \subset Z^d$ is defined by

$$d(G) = \sup\{r\colon\ \exists\ c \text{ such that } B(r,c) \cap Z^d \subset G\}.$$

Here $B(r,c)$ denotes the ball of radius r and center c with respect to $\|x\|_\infty = \max_{1\leqslant j\leqslant m}|x_j|$.

Let $\tau_j,\ j\in Z^d$, be the translation operator defined on $(R)^{Z^d}$ by $\tau_j(\xi)=\{\xi_{i+j},\ i\in Z^d\}$. Denote by $\mathcal{A}$ the σ-algebra of invariant sets of $(R)^{Z^d}$: a Borel set $A\in\mathcal{A}$ if and only if $\tau_j(A)=A$ for all $j\in Z^d$. By $\mathcal{T}=\xi^{-1}(\mathcal{A})$ we denote the σ-algebra of invariant sets of the field ξ. The random field ξ is called ergodic if its σ-algebra $\mathcal{T}$ is trivial. For example, if $\xi=\{\xi_i,\ i\in Z^d,\}$ are i.i.d. random variables, then ξ a is stationary ergodic random field. If $\eta=\{\eta_i=g(\tau_i(\xi)),\ i\in Z^d\}$, where g is a measurable mapping from $(R)^{Z^d}$ to any measurable space and if ξ is ergodic, then η also is ergodic. The following result is the law of large numbers for ergodic random fields.

THEOREM C [Theorem 3.1.1 in [6]]. *Let $X=\{X_i, i\in\mathbf{Z}^d\}$ be a stationary real-valued random field with $E|X_i|^p<\infty$ for some $1\leqslant p<\infty$.*

(i) *Let $G_n\subset\mathbf{Z}^{\mathbf{d}}$ be a sequence of convex and bounded sets such that $d(G_n)\to\infty$. Then*

$$|G_n|^{-1}\sum_{i\in G_n}X_i\xrightarrow{L^p}E(X|\mathcal{I}).\tag{6}$$

Here $\mathcal{I}$ is a translation invariant σ-algebra of the process X.

(ii) *If additionally $G_n\subset G_{n+1}$, then convergence in* (6) *is a.s.*

In our proofs, we need the following lemma, which is a corollary of Theorem C.

LEMMA 1 [Corollary 3.1.1 in [6]]. *Let $X=\{X_i, i\in\mathbf{Z}^d\}$ be an ergodic stationary field, and let g: $\mathbf{R}^{\mathbf{Z}^d}\to\mathbf{R}$ be an integrable function.*

(a) *If G_n is a sequence of bounded convex sets such that $d(G_n)\to\infty$, then*

$$|G_n|^{-1}\sum_{i\in G_n}g(X\circ\tau_i)\xrightarrow{L^1}E(g(X)).$$

(b) *If the sequence $(G_n, n\geqslant 1)$ is increasing, then the limit relation holds a.s.*

The next two lemmas were proved in [4].

LEMMA 2 [Lemma 2.1 in [4]]. *Let $r,k,n,m_1,\dots,m_r$ be positive integers such that $1\leqslant r\leqslant k\leqslant n,\ m_i\geqslant 1$ for all i, and $m_1+\ldots+m_r=k$. Define $n_r=\left[\frac{n}{r}\right]$ and $s=\#\{i\leqslant r\colon\ m_i=1\}$. Then, for any set of i.i.d. random variables $\xi_i,\ i\leqslant n$, the following inequality holds:*

$$n_r^r\binom{k}{m_1,\dots,m_r}^{1/2}\left|E\frac{\xi_1^{m_1}\dots,\xi_r^{m_r}}{(\sum_{1\leqslant i\leqslant n}\xi_i^2)^{k/2}}\right|\leqslant\left(E\left|\frac{\sum_1^{n_r}\xi_i}{(\sum_{1\leqslant i\leqslant n_r}\xi_i^2)^{1/2}}\right|\right)^s.$$

LEMMA 3. [4]. *Let* $(\xi_j,\ j \in \mathbf{Z})$ *be a sequence of i.i.d. r.v's with* $E\xi_1 = 0$ *and* $\xi_1 \in DAN$. *Then the sequence*

$$\Big(\sum_{1\leqslant i\leqslant n} \xi_i\Big)\Big(\sum_{1\leqslant i\leqslant n} \xi_i^2\Big)^{-1/2}, \quad n \geqslant 1,$$

is stochastically bounded and

$$\sup_n E\Big|\frac{\sum_{1\leqslant i\leqslant n}\xi_i}{(\sum_{1\leqslant i\leqslant n}\xi_i^2)^{1/2}}\Big| < \infty.$$

Here the abbreviation $X \in DAN$ means that X belongs to the domain of attraction of normal law. Lemmas 2 and 3 are important tools in the proof of our results. We also use the following theorem from [1].

THEOREM D [Proposition 1.3 in[1]]. *Suppose that* $X_{t,s}$ *is the process defined by* (2) *with* $|a| + |b| < 1$. *Then*

$$(M_1M_2)^{-1/2} \sum_{(t,s)\in D_n} \begin{pmatrix} \epsilon_{t,s}X_{t-1,s} \\ \epsilon_{t,s}X_{t,s-1} \end{pmatrix} \xrightarrow{\mathcal{D}} N(0, \Sigma_{a,b}),$$

where $N(0, \Sigma_{a,b})$ *denotes the two-dimensional normal law with mean zero and the covariance matrix*

$$\Sigma_{a,b} = \begin{pmatrix} \gamma_{(0,0)} & \gamma_{(-1,1)} \\ \gamma_{(-1,1)} & \gamma_{(0,0)} \end{pmatrix}.$$

The quantities $\gamma_{(0,0)}$ and $\gamma_{(-1,1)}$ were defined before the formulation of Theorem 1. The original formulation of Theorem D in [1] was slightly different from ours. The result in [1] was actually proved for triangular domains $T_n = \{t, s\colon\ 0 \leqslant t + s,\ \ \max(t, s) \leqslant n\}$; however, it is easy to see that the same result can be proved for the domains D_n considered in this paper. One would have to follow the original proof and make some minor changes.

3. PROOF OF THEOREMS

We start the proof of Theorem 1 by introducing more notation:

$$\chi_n^2 := \sum\nolimits^* \epsilon_{t,s}^2,\ \ R_n := |D_n| = M_{n,1}M_{n,2},$$
$$Z_n := \sum\nolimits^* X_{t,s},\ \ F_n^2 := \sum\nolimits^* X_{t,s}^2.$$

Using (2), we have

$$Z_n = a\sum\nolimits^* X_{t-1,s} + b\sum\nolimits^* X_{t,s-1} + \sum\nolimits^* \epsilon_{t,s}. \tag{7}$$

Since

$$Z_n = \sum\nolimits^* X_{t-1,s} + \sum_{s=1}^{M_2} X_{M_1,s} - \sum_{s=1}^{M_2} X_{0,s},$$

$$Z_n = \sum\nolimits^* X_{t,s-1} + \sum_{t=1}^{M_1} X_{t,M_2} - \sum_{t=1}^{M_1} X_{t,0},$$

we easily obtain

$$Z_n = (1-a-b)^{-1} \sum\nolimits^* \epsilon_{t,s} - (1-a-b)^{-1} R_n^{(1)}. \tag{8}$$

Then we get

$$F_n^{-1} Z_n = C_n (A_n + R_n^{(1)} \chi_n^{-1}),$$

where

$$A_n = \frac{\sum^* \epsilon_{t,s}}{\chi_n},$$

$$R_n^{(1)} = b \sum_{t=1}^{M_1} X_{t,M_2} + a \sum_{s=1}^{M_2} X_{M_1,s} - a \sum_{s=1}^{M_2} X_{0,s} - b \sum_{t=1}^{M_1} X_{t,0},$$

and

$$C_n^2 = (1-a-b)^{-2} \chi_n^2 F_n^{-2}.$$

From the classical theorem for the self-normalized sequences we know that $A_n \xrightarrow{\mathcal{D}} N(0,1)$. We will show that $\chi_n^{-1} R_n^{(1)} \xrightarrow{P} 0$ and $C_n \xrightarrow{P} \sigma(a,b)$. From these three relations we shall get (4).

The proof of the fact $R_n^{(1)} \chi_n^{-1} \xrightarrow{P} 0$ consists of two parts: the first one is the classical law of large numbers,

$$\frac{\chi_n^2}{R_n} \xrightarrow{a.s.} 1, \tag{9}$$

and the second one is the relation

$$R_n^{-1/2} R_n^{(1)} \xrightarrow{P} 0. \tag{10}$$

To prove (10) we use the Chebyshev inequality: for all $h > 0$,

$$P\left(|R_n^{(1)}| > h R_n^{1/2}\right) \leqslant h^{-2} E\left(R_n^{(1)}\right)^2 R_n^{-1}.$$

We have

$$E(R_n^{(1)})^2 = K_1 + K_2 + K_3 + K_4,$$

where

$$K_1 = b^2 E\Big(\sum_{t=1}^{M_1} X_{t,M_2}\Big)^2 + b^2 E\Big(\sum_{t=1}^{M_1} X_{t,0}\Big)^2$$
$$+ a^2 E\Big(\sum_{s=1}^{M_2} X_{M_1,s}\Big)^2 + a^2 E\Big(\sum_{s=1}^{M_2} X_{0,s}\Big)^2,$$
$$K_2 = +2ab \sum_{t=1}^{M_1}\sum_{s=1}^{M_2} EX_{t,M_2}X_{M_1,s} + 2ab \sum_{t=1}^{M_1}\sum_{s=1}^{M_2} EX_{0,s}X_{t,0},$$
$$K_3 = -2ab \sum_{t=1}^{M_1}\sum_{s=1}^{M_2} EX_{t,M_2}X_{0,s} - 2ab \sum_{t=1}^{M_1}\sum_{s=1}^{M_2} EX_{t,0}X_{M_1,s},$$
$$K_4 = -2b^2 \sum_{t=1}^{M_1}\sum_{t_1=1}^{M_1} EX_{t,M_2}X_{t_1,0} - 2a^2 \sum_{s=1}^{M_2}\sum_{s_1=1}^{M_2} EX_{M_1,s}X_{0,s_1}.$$

Having stationary solution (3), it is easy to estimate the covariances of the process $X_{t,s}$ and we use such estimates from [1]:

$$|\gamma_{(l_1,l_2)}| \leqslant (1 - (|a| + |b|)^{-2})^{-1}(|a| + |b|)^{|l_1|+|l_2|} \tag{11}$$

This estimate of the covariances together with the stationarity of the process allows us to get the following bounds for all K_i:

$$|K_1| \leqslant C(4b^2 M_1 + 4a^2 M_2),$$
$$|K_2| \leqslant C, \quad |K_3| \leqslant C,$$
$$|K_4| \leqslant C(4b^2(|a| + |b|)^{M_2} M_1 + 4a^2(|a| + |b|)^{M_1} M_2).$$

From these estimates we get (10).

The last step in the proof of Theorem 1 is the estimation of C_n.

We can rewrite C_n as

$$C_n^2 = (1 - a - b)^2 \chi_n^2 R_n^{-1} F_n^{-2} R_n. \tag{12}$$

We have (9), therefore, it remains to find a limit for $F_n^{-2} R_n$.

We split the quantity F_n^2 into four parts:

$$F_n^2 = F_n^{(1)} + F_n^{(2)} + R_n^{(2)} + \sum{}^* \epsilon_{t,s}^2,$$

where

$$\begin{aligned} F_n^{(1)} &= a^2 \sum{}^* X_{t-1,s}^2 + b^2 \sum{}^* X_{t,s-1}^2, \\ F_n^{(2)} &= 2ab \sum{}^* X_{t-1,s} X_{t,s-1}, \\ R_n^{(2)} &= 2a \sum{}^* X_{t-1,s} \epsilon_{t,s} + 2b \sum{}^* X_{t,s-1} \epsilon_{t,s}. \end{aligned}$$

Using the obvious identities

$$\sum{}^* X_{t,s}^2 = \sum{}^* X_{t-1,s}^2 + \sum_{s=1}^{M_2} X_{M_1,s}^2 - \sum_{s=1}^{M_2} X_{0,s}^2,$$

$$\sum{}^* X_{t,s}^2 = \sum{}^* X_{t,s-1}^2 + \sum_{t=1}^{M_1} X_{t,M_2}^2 - \sum_{t=1}^{M_1} X_{t,0}^2,$$

we get

$$F_n^{(1)} = (a^2 + b^2) \sum{}^* X_{t,s}^2 + R_n^{(3)}$$

and

$$F_n^2 = (1 - a^2 - b^2)^{-1} \left(F_n^{(2)} + R_n^{(2)} + R_n^{(3)} + \sum{}^* \epsilon_{t,s}^2 \right). \tag{13}$$

Here

$$R_n^{(3)} = b^2 \sum_{t=1}^{M_1} X_{t,0}^2 + a^2 \sum_{s=1}^{M_2} X_{0,s}^2 - b^2 \sum_{t=1}^{M_1} X_{t,M_2}^2 - a^2 \sum_{s=1}^{M_2} X_{M_1,s}^2.$$

Due to the stationarity of the process $X_{t,s}$ we can write

$$E \sum_{t=1}^{M_1} X_{t,2}^2 = M_{n,1} \gamma_{(0,0)};$$

therefore,

$$R_n^{-1} \sum_{t=1}^{M_1} X_{t,M_2}^2 \xrightarrow{P} 0.$$

Since all the other summands in $R_n^{(3)}$ can be dealt in the same way, we get

$$R_n^{-1} R_n^{(3)} \xrightarrow{P} 0. \tag{14}$$

Setting $g(X) = X_{0,0}X_{-1,1}$, from Lemma 1 we find

$$R_n^{-1}F_n^{(2)} \xrightarrow{P} 2ab\gamma_{(-1,1)}. \tag{15}$$

Theorem D implies that $R_n^{-1/2}R_n^{(2)} \xrightarrow{D} N(0, \theta_{a,b})$. The expression of the variance $\theta_{a,b} = (4a^2 + 4b^2)\gamma_{(0,0)} + 8ab\gamma_{(-1,1)})$ can be also found from Theorem D. Therefore,

$$R_n^{-1}R_n^{(2)} \xrightarrow{P} 0. \tag{16}$$

Collecting the obtained relations (12)–(16), we get

$$C_n \xrightarrow{P} \sigma(a, b).$$

Theorem 1 is proved.

Proof of Theorem 2 uses some steps of the proof of Theorem 1. We want to prove the relation

$$\frac{\sum_* Y_{i,j}}{\left(\sum_* Y_{i,j}^2\right)^{1/2}} \xrightarrow{\mathcal{D}} N(0, 1). \tag{17}$$

The nominator in (17) is simply Z_n and, since we already have relations (8) and (10), it is easy to see that, in order to prove (17), we need to show that

$$\frac{\sum_* Y_{i,j}^2}{R_n} \xrightarrow{P} (1 - a - b)^{-2}. \tag{18}$$

Similarly to relation (7), using the definition of the process (2), we can write

$$Y_{i,j} = \zeta_{i,j} + \kappa_{i,j} + \eta_{i,j}$$

and

$$Y_{i,j}^2 = \zeta_{i,j}^2 + \kappa_{i,j}^2 + \eta_{i,j}^2 + 2(\zeta_{i,j}\kappa_{i,j} + \zeta_{i,j}\eta_{i,j} + \kappa_{i,j}\eta_{i,j}). \tag{19}$$

Here

$$\zeta_{i,j} := (1 - a - b)^{-1} \sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{s=m_{2,j}+1}^{m_{2,j+1}} \epsilon_{t,s},$$

$$\kappa_{i,j} := b(1 - a - b)^{-1} \sum_{t=m_{1,i}+1}^{m_{1,i+1}} (X_{t,m_{2,j}} - X_{t,m_{2,j+1}}),$$

$$\eta_{i,j} := a(1-a-b)^{-1} \sum_{s=m_{2,j}+1}^{m_{2,j+1}} (X_{m_{1,i},s} - X_{m_{1,i+1},s}).$$

We will show that the main input into sum (19) is given by the term $\zeta_{i,j}^2$, while all the remaining terms are negligible. From the definition of $\zeta_{i,j}$ we have the equality

$$(1-a-b)^2 \zeta_{i,j}^2 = \sum_{(t,s)\in D_{i,j}} \epsilon_{t,s}^2 + \sum_{(t,s)\in D_{i,j}} \sum_{\substack{(t_1,s_1)\in D_{i,j} \\ (t,s)\neq(t_1,s_1)}} \epsilon_{t,s}\epsilon_{t_1,s_1}. \tag{20}$$

Lemmas 2 and 3 give the following estimate for the expectations of summands in (20) with $(t,s) \neq (t_1,s_1)$:

$$\left| E \frac{\epsilon_{t,s}\epsilon_{t_1,s_1}}{\chi_n^2} \right| \leqslant C R_n^{-2}.$$

Simple calculations lead to the estimate of the expectation

$$\sum\nolimits_* \left| E\chi_n^{-2} \sum_{(t,s)\in D_{i,j}} \sum_{\substack{(t_1,s_1)\in D_{i,j} \\ (t,s)\neq(t_1,s_1)}} \epsilon_{t,s}\epsilon_{t_1,s_1} \right| \leqslant C I J (m_1 m_2)^2 R_n^{-2} \leqslant C \frac{m_1 m_2}{R_n}.$$

This estimate gives us

$$\sum\nolimits_* \sum_{(t,s)\in D_{i,j}} \sum_{\substack{(t_1,s_1)\in D_{i,j} \\ (t,s)\neq(t_1,s_1)}} \frac{\epsilon_{t,s}\epsilon_{t_1,s_1}}{\chi_n^2} \xrightarrow{P} 0, \tag{21}$$

and from (20) and (21), taking into account (9), we get

$$R_n^{-1} \sum_{i,j} \zeta_{i,j}^2 \xrightarrow{P} (1-a-b)^{-2}. \tag{22}$$

Now we estimate the remaining terms in (19). From the definition of $\kappa_{i,j}$ we have

$$\begin{aligned}(1-a-b)^2 b^{-2} \kappa_{i,j}^2 = &\sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{t_1=m_{1,i}+1}^{m_{1,i+1}} X_{t,m_{2,j}} X_{t_1,m_{2,j}} \\ &- 2 \sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{t_1=m_{1,i}+1}^{m_{1,i+1}} X_{t,m_{2,j}} X_{t_1,m_{2,j+1}}\end{aligned}$$

$$+ \sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{t_1=m_{1,i}+1}^{m_{1,i+1}} X_{t,m_{2,j+1}} X_{t_1,m_{2,j+1}}.$$

The stationarity of the process $X_{t,s}$ ensures that the expectation $E\kappa_{i,j}^2$ is estimated by the sums of covariances:

$$\Big| E \sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{t_1=m_{1,i}+1}^{m_{1,i+1}} X_{t,m_{2,j}} X_{t_1,m_{2,j}} \Big| \leqslant 2m_1 \sum_{k=0}^{m_1} |\gamma_{k,0}|,$$

$$\Big| E \sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{t_1=m_{1,i}+1}^{m_{1,i+1}} X_{t,m_{2,j}} X_{t_1,m_{2,j+1}} \Big| \leqslant \sum_{t=1}^{m_1} \sum_{t_1=0}^{m_1} |\gamma_{t-t_1,m_2}|,$$

$$\Big| E \sum_{t=m_{1,i}+1}^{m_{1,i+1}} \sum_{t_1=m_{1,i}+1}^{m_{1,i+1}} X_{t,m_{2,j+1}} X_{t_1,m_{2,j+1}} \Big| \leqslant 2m_1 \sum_{k=0}^{m_1} |\gamma_{k,0}|.$$

Using estimates of covariances (11) in the same way as in estimating quantities $K_i,\ i = 1, \dots, 4$, we easily get

$$R_n^{-1} E \textstyle\sum_* \kappa_{i,j}^2 \leqslant CIJm_1 R_n^{-1} \leqslant Cm_2^{-1}.$$

From this it follows that

$$R_n^{-1} \textstyle\sum_* \kappa_{i,j}^2 \xrightarrow{P} 0, \tag{23}$$

and in a similar way one can get

$$\frac{\sum_* \eta_{i,j}^2}{R_n} \xrightarrow{P} 0. \tag{24}$$

Having relations (22), (23), and (24), the remaining sums of products of $\zeta_{i,j},\ \eta_{i,j}$, and $\kappa_{i,j}$ are estimated using the Cauchy inequality; for example,

$$\big| \textstyle\sum_* R_n^{-1} \zeta_{i,j} \kappa_{i,j} \big| \leqslant C \left(\sum_* \zeta_{i,j}^2 R_n^{-1} \right)^{1/2} \left(\sum_* \kappa_{i,j}^2 R_n^{-1} \right)^{1/2}.$$

From (22)and (23) we get

$$\big| R_n^{-1} \textstyle\sum_* \zeta_{i,j} \kappa_{i,j} \big| \xrightarrow{P} 0. \tag{25}$$

The same relation holds for other two sums. From (22), (23), (24), and (25) relation (18) follows, and Theorem 2 is proved.

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REZIUMĖ

V. Paulauskas, R. Zovė. Pastaba apie paprasto erdvinio autoregresinio modelio autonormavimą

Straipsnyje nagrinėjamas gana paprasto autoregresinio modelio $X_{t,s} = aX_{t-1,s} + bX_{t,s-1} + \varepsilon_{t,s}$, apibrėžto dvimatėje sveikų skaičių gardelėje, autonormavimas. Įrodomos dvi teoremos, apibendrinančios analogiškus [8] darbo rezultatus autoregresinio proceso $AR(1)$ atveju.