ON OPERATOR-NORM APPROXIMATION OF SOME SEMIGROUPS BY QUASI-SECTORIAL OPERATORS

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ABSTRACT

We use some results and methods of probability theory to improve bounds for the convergence rates in some approximation formulas for operators.

The aim of this note is to demonstrate that some results and methods of probability theory can be useful in a particular field of operator theory - convergence rates in the approximation formulas for some semigroups of operators.

Let $\mathbf{H}$ be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and the norm $||x|| = (x,x)^{1/2}$, and let T be a linear operator in $\mathbf{H}$. Let $\mathbb{C}$, as usual, denote the complex plane. Consider the sets (cf. [2])

$$\Theta(T) = \{(Tx,x), x \in \mathcal{D}(T), ||x|| = 1\},$$

where $\mathcal{D}(T)$ is the domain of T,

$$S_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\},\ 0 < \alpha \le \pi,$$

and, for $0 \le \alpha \le \pi/2$,

$$D_\alpha = \{z \in \mathbb{C} : |z| \le \sin\alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \le \alpha,\ |z-1| \le \cos\alpha\}.$$

An Operator T is called sectorial with semi-angle $\alpha \in (0,\pi/2)$ and vertex at 0 if $\Theta(T) \subseteq S_\alpha$. If, in addition, T is closed and there exists $z \notin S_\alpha$ belonging to the resolvent set of T, then T is said to be m-sectorial.

Let A be a contraction on $\mathbf{H}$. We say that A is a quasi-sectorial operator with a semi-angle $\alpha \in [0,\pi/2)$ with respect to the vertex at 1 if $\Theta(A) \subset D_\alpha$.

Quasi-sectorial operators were introduced in [2], where , among other results, the following extension of the famous Chernoff "$n^{1/2}$- lemma" (see [6], Lemma 2) was proved: .

Theorem A([2]). *Let A be a quasi-sectorial contraction on* $\mathbf{H}$ *with numerical range* $\Theta(A) \subset D_\alpha$, $\alpha \in [0, \pi/2)$. *Then*

$$\|A^n - e^{n(A-I)}\| \le 2(K+1)n^{-1/3}, \ n = 1, 2, \dots, \tag{1}$$

where K is a constant depending on α (its explicit expression is given in [2]).

The quantity estimated in this theorem can be written as (see formula (14) in [2])

$$I(n) = \|A^n - e^{n(A-I)}\| = \|e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!}(A^n - A^k)\|.$$

If X is a Poisson random variable with mean $EX = n$, then

$$I(n) = \|E(A^n - A^X)\|,$$

where E denotes mathematical expectation, and A^X is an operator-valued random variable taking values A^k with probabilities $P\{X = k\} = e^{-n}n^k(k!)^{-1}$. If we use the above written representation of $I(n)$, then we need to estimate $\|A^n - A^k\|$ and Poissonian tail probabilities $P\{|X - n| > k\}$. Estimates of these probabilities used in the proof of (1) were too rough, and it was not difficult to note that, using the same bounds for $\|A^n - A^k\|$ as in [2] but more precise bounds for tail probabilities, we can improve estimate (1). Namely, the following result holds.

Theorem 1. *Under the conditions of Theorem A, we have the bound*

$$\|A^n - e^{n(A-I)}\| \le K_1 \Big(\frac{\ln n}{n}\Big)^{1/2}, \tag{2}$$

where the constant K_1 depends on α.

Proof of Theorem 1. We begin the proof by formulating a bound on Poisson tail probabilities following from [7](see also [1], where an estimate of the same precision was obtained).

Proposition 2. [7]. *Let X be a Poisson random variable with parameter $\lambda > 0$. Then, for all $x > 0$, we have*

$$\max\Big(P\Big\{X \le \lambda - x\Big\}, P\Big\{X \ge \lambda + x\Big\}\Big) \le \exp\Big\{-(x+\lambda)\log\big(1 + \frac{x}{\lambda}\big) + x\Big\}. \tag{3}$$

In this estimate, the exponent is a right one. In [10], it was shown that exact (up to constants) estimates from above and bellow contain the additional term $(\lambda + x)^{-1/2}$ but these exact estimates for tail probabilities were proved only for $x > \lambda - 1$. Since we need to estimate tail probabilities when x is of order $n^{1/2}$ and, in our case, $\lambda = n$, we cannot apply the exact estimate from [10].

Now we can prove (2). We choose $k = [(6n\ln n)^{1/2}]$ (as usual, $[a]$ denotes the integer part of a number $a > 0$) and divide $I(n)$ into two parts:

$$I(n) \le I_1(n) + I_2(n), \tag{4}$$

where

$$I_1(n) = \sum_{j\ge 0, |j-n|\ge k} p_j(n)\|A^n(I-A^j)\|, \qquad I_2(n) = \sum_{j=n-k}^{n+k} p_j(n)\|A^n(I-A^j)\|,$$

and $p_j(n) = e^{-n}n^j(j!)^{-1}$. In the first sum, estimating $\|A^n(I-A^j)\| \le 2$ and using (3), we have

$$I_1(n) \le 2\Big(P\Big\{X \le n-k\Big\} + P\Big\{X \ge n+k\Big\}\Big) \le$$

$$\le 4\exp\Big\{-(n+k)\log\big(1+\frac{k}{n}\big) + k\Big\} \le 4\exp\Big\{-\frac{k^2}{2n}(1-k/n)\Big\} \le \frac{4}{n} \tag{5}$$

for $n > n_0$, where n_0 is such that $n_0^{-1}\ln n_0 \le 2/27$. Here we used the elementary inequality $\ln(1+x) \ge x - x^2/2$ for $0 < x < 1$.

To estimate the second sum we use the following estimate from [2].

Lemma 3.([2]). *If A is a quasi-sectorial contraction on $\mathbf{H}$ with semi-angle $\alpha \in [0, \pi/2)$, then*

$$\|A^n(I-A)\| \le \frac{K_2}{n+1}, \tag{6}$$

for all $n \ge 1$, where K_2 is a constant depending on α.

Using a telescoping sum, we get, for every integer $k \ge 1$,

$$\|A^n(I-A^k)\| \le \frac{K_2 k}{n}. \tag{7}$$

Therefore, if $n < j < n+k$, then

$$\|A^n - A^j\| = \|A^n(I-A^{j-n})\| \le \frac{K_2 k}{n}$$

and if $n-k \le j < n$ then

$$\|A^n - A^j\| = \|A^j(I-A^{n-j})\| \le \frac{K_2(n-j)}{j} \le \frac{K_2 k}{n-k} \le \frac{3K_2 k}{n}$$

for $n > n_0$. Thus, for $n > n_0$, we have

$$I_2(n) \le 3\sqrt{6}K_2\Big(\frac{\ln n}{n}\Big)^{1/2}. \tag{8}$$

From (4), (5), and (8), for $n > n_0$, we have

$$I(n) \le (3\sqrt{6}K_2 + 4)\Big(\frac{\ln n}{n}\Big)^{1/2}.$$

Therefore, taking $K_1 = \max(n_0, (3\sqrt{6}K_2 + 4))$, we have (2) for all n, and Theorem 1 is proved.

It remains an open question what is the optimal rate of convergence in Theorem 1. One can suspect that estimate (7) obtained by using telescoping sums is too rough but it turns out that it is not the case. It is easy to verify that, for $1 \le k \le n$,

$$\sup_{0\le a\le 1} a^n(1-a^k) = \Big(\frac{n}{n+k}\Big)^{n/k}\frac{k}{n+k},$$

therefore we cannot improve (7). This fact together with the precision of (3) make us to believe that it is impossible to get the rate of convergence better than $n^{-1/2}$ using the approach based on Poisson probabilities. It is worth to mention that this approach comes from Chernoff paper [6].

In Theorem 1 estimate (1) was improved from $n^{-1/3}$ to the rate $(n^{-1}\ln n)^{1/2}$. The next result presents a smaller improvement, in which we take off the logarithmic term in another error estimate from [2]. Despite of this we think that our Theorem 4 is more important, since it gives an optimal estimate and, in its proof, a new idea (induction method) is used. The proof of Theorem 4 also indicates a possible way to estimate the rate of convergence in general Chernoff theorem (see Proposition 7 bellow).

Now we consider the following operator-norm Euler approximation of the exponential function proved in [2].

Theorem B. *If A is an m-sectorial operator in a Hilbert space* **H** *with semi-angle $\alpha \in (0, \pi/2)$ and vertex at 0, then, for every $t \in S_{\pi/2-\alpha}$,*

$$\lim_{n\to\infty} \|(I + tA/n)^{-n} - e^{-tA}\| = 0. \tag{9}$$

Moreover, if 0 belongs to the resolvent set of A, then, uniformly in $t \ge 0$,

$$\|(I + tA/n)^{-n} - e^{-tA}\| = O\Big(\frac{\ln n}{n}\Big). \tag{10}$$

This result resembles the Central Limit Theorem (CLT) in probability theory. More precisely, considering e^{-tA} as a Gaussian distribution (in some sense, it is a “nice” operator) and $(I + tA/n)^{-n}$ as the distribution of a normalized sum of n independent identically distributed summands, one can interpret (9) as the CLT and (10) as the rate of convergence in this theorem. It also turns out that, using the same estimates of operators from [2] but applying the method of induction, which is commonly used to get the rates of convergence in the CLT (especially in infinite-dimensional spaces where Fourier analysis is not so successful as in finite-dimensional spaces, see [11] and references therein) we are able to improve (10) to the optimal bound.

Theorem 4. *Under the conditions of Theorem B (for (10)), we have*

$$\sup_{t\ge 0} \|(I + tA/n)^{-n} - e^{-tA}\| \le \bar{C}n^{-1}, \tag{11}$$

where $\bar{C}$ is a constant depending on the operator A.

Proof of Theorem 4. Let us denote

$$\delta_n(t) = (I + tA/n)^{-n} - e^{-tA}, \quad \Delta_n(t) = \|\delta_n(t)\|, \quad \Delta_n = \sup_{t\geq 0} \|\Delta_n(t)\|.$$

We must prove that, for all $n \geq 1$,

$$\Delta_n \leq \bar{C} n^{-1}. \tag{12}$$

Since the operators $F(t) = (I + tA)^{-1}$ are quasi-sectorial contractions (see Section 2.1 in [2]) and the operator A generates a holomorphic semigroup (see Proposition 1.4 of [2]), we have that

$$\Delta_1 \leq 2,$$

that is, (12) holds for $n = 1$ with $\bar{C} \geq 2$. Now let us suppose (this is the main step of induction) that, for all $1 < k \leq n - 1$,

$$\Delta_k \leq \bar{C} k^{-1}. \tag{13}$$

We shall prove that (13) also holds for $k = n$. We assume that $n > n_0$ (later we shall put some condition on n_0), since, for $n \leq n_0$, (12) trivially holds with $\bar{C} \geq 2n_0$. We first collect all estimates used to prove (10) in one lemma ; the proofs of these estimates can be found in [2] and [3]. In what follows, C_A denote constants, depending only on operator A and, possibly, different in different places.

Lemma 5. ([2], [3]). *Under the conditions of Theorem 4, the following estimates hold:*

$$\|A^{-1}((I + tA/n)^{-1} - e^{-tA/n})\| \leq 2t/n, \tag{14}$$

$$\|A^{-1}((I + tA/n)^{-1} - e^{-tA/n})A^{-1}\| \leq 3/2(t/n)^2, \tag{15}$$

$$\|((I + tA/n)^{-1} - e^{-tA/n})A^{-2}\| \leq 3/2(t/n)^2, \tag{16}$$

$$\|A^i e^{-\tau A}\| \leq C_A \tau^{-i}, i = 1, 2, \quad \tau > 0, \tag{17}$$

$$\|(I + tA/n)^{-k} A\| \leq nK_2/kt, \tag{18}$$

where K_2 is from (6).

Along with the notation $F(t) = (I + tA)^{-1}$, let us denote $G(t) = e^{-tA}$. Since $\delta_n(t) = F(t/n)^n - G(t/n)^n$, we can write the well-known identity

$$\delta_n(t) = \sum_{k=0}^{n-1} J_k(t), \tag{19}$$

where

$$J_k(t) = \big(F\big(t/n\big)\big)^{n-1-k} \big((F\big(t/n\big) - G(\big(t/n\big)\big)\big(G\big(t/n\big)\big)^k.$$

Denoting $J_k = \sup_{t\geq 0} \|J_k(t)\|$, from (19) we get

$$\Delta_n \leq \sum_{k=0}^{n-1} J_k. \tag{20}$$

We first separate the last term, which is easily estimated using (14) and (17) with $i=1$:

$$J_{n-1} \le \frac{2C_A}{n-1} \le 4C_A n^{-1}. \tag{21}$$

Let C_1 be some positive integer, which we shall specify later. We divide the rest of the sum in (20) (without J_{n-1}) into two parts: $\Delta_n - J_{n-1} = \Delta_n^{(1)} + \Delta_n^{(2)}$, where

$$\Delta_n^{(1)} = \sum_{k=0}^{C_1} J_k, \quad \Delta_n^{(2)} = \sum_{k=C_1}^{n-2} J_k.$$

In the first sum, we estimate $\|G((t/n)\| \le 1$ and then use (18) and (14). We get

$$J_k \le 2K_2(n-1-k)^{-1}$$

and, therefore,

$$\Delta_n^{(1)} = \sum_{k=0}^{C_1} \frac{2K_2}{n-1-k} \le \frac{2C_1K_2}{n-1-C_1} \le \frac{4C_1K_2}{n} \tag{22}$$

(here we made the first assumption about C_1: $C_1 \le n/2-1$). In the second sum, we estimate $J_k \le J_k^1 + J_k^2$, where $J_k^i = \sup_{t\ge 0}\|J_k^i(t)\|$, $i=1,2$, and

$$J_k^1(t) = \Big(\big(F(t/n)\big)^{n-1-k} - \big(G(t/n)\big)^{n-1-k}\Big)\big(F(t/n) - G((t/n)\big)\big(G(t/n)\big)^k,$$

$$J_k^2(t) = \big(G(t/n)\big)^{n-1-k}\Big((F(t/n) - G((t/n)\Big)\big(G(t/n)\big)^k.$$

Note that

$$\big(F(t/n)\big)^{n-1-k} - \big(G(t/n)\big)^{n-1-k} = \Big(F(\frac{\tau}{n-1-k})\Big)^{n-1-k} - \Big(G(\frac{\tau}{n-1-k})\Big)^{n-1-k}$$

with $\tau = t(n-1-k)n^{-1}$ and, therefore,

$$\sup_{t\ge 0}\Big\|\Big(F(t/n)\Big)^{n-1-k} - \Big(G(t/n)\Big)^{n-1-k}\Big\| = \sup_{\tau\ge 0}\|\Delta_{n-1-k}(\tau)\| = \Delta_{n-1-k}.$$

For the term

$$\|\big(F(t/n) - G((t/n)\big)\big(G(t/n)\big)^k\|,$$

applying (16) and (17) with $i=2$ and using assumption (13), we get

$$J_k^1 \le \Delta_{n-1-k}\frac{3t^2C_An^2}{2n^2t^2k^2} \le \frac{3C_A\bar{C}}{2k^2(n-1-k)}.$$

Using again (16) and (17) with $i = 2$ and separately considering the cases $k > n/2$ and $k \leq n/2$, we easily get

$$J_k^2 \leq \frac{6C_A}{(n-2)^2}.$$

It is easy to verify that the following bound holds:

$$\sum_{k=C_1}^{n-2} \frac{1}{(n-1-k)k^2} \leq \frac{12}{nC_1} + \frac{4\ln n}{n^2}.$$

Therefore, from two previous estimates we easily get

$$\Delta_n^{(2)} \leq \sum_{k=C_1}^{n-2} J_k^1 + \sum_{k=C_1}^{n-2} J_k^2 \leq$$

$$\leq \frac{3C_A\bar{C}}{2}\Big(\frac{12}{nC_1} + \frac{4\ln n}{n^2}\Big) + \frac{6C_A}{n-2}. \tag{23}$$

Collecting (20)–(23), we get

$$\Delta_n \leq \frac{\bar{C}}{n}\Big(C_A\Big(\frac{1}{C_1} + \frac{\ln n}{n}\Big) + \frac{C_A(1+C_1)}{\bar{C}}\Big).$$

Now we choose C_1 and n_0 such that, for all $n > n_0$,

$$C_A\Big(\frac{1}{C_1} + \frac{\ln n}{n}\Big) < \frac{1}{2} \tag{24}$$

and then take $\bar{C}$ satisfying

$$\bar{C} > \max\{2n_0, 2C_A(1+C_1)\}.$$

Then we have

$$\Delta_n \leq \bar{C}n^{-1},$$

and the theorem is proved.

There are more results on convergence of operators in the Trotter product formulae (see, for example, Theorems 3.4 and 5.8 of [4], Theorem 1 of [3], Theorem 3.5 of [5]) containing a logarithmic term in estimates. We hope that applying the induction method, as demonstrated above, it is possible to get rid of this logarithmic term in all these estimates.

(**Note added during revision**. In a recent preprint of V. Cachia "Euler's exponential formula for non C_0 semigroups" (available at http:// mpej.unige.ch/ ˜ cachia, to appear in Semigroup Forum (2003)) the rate $O(\frac{\ln n}{n})$ in Theorem B is extended to bounded holomorphic semigroups not necessarily of class C_0 and at the end of the preprint it is claimed that induction method allows to skip the factor $\ln n$.)

In our preprint [12], we discussed in detail similarities and differences in estimating the rate of convergence in the CLT in probability theory and in relation (9) for operators. Here we only note that the main difference is that, in the CLT, we use the so-called "smoothing" inequality. This means that, at the beginning of the proof, we take the convolution of the difference of distribution of a sum and a limiting Gaussian distribution with a Gaussian law with small variance. Since, in our context, the role of Gaussian distribution is played by the operator $\exp(-tA)$, it is temptating to use the operator $G(1/n) = \exp(-A/n)$ as a "smoothing" factor. But it turns out that the family of operators $\exp(-tA)$ is not so good as Gaussian distributions (due to the fact that A is an unbounded operator) and at present we cannot provide any reasonable "smoothing" inequality. It seems that it is interesting to find an appropriate "smoothing" inequality for the operator convergence.

Theorem B and Theorem 4 give the rate of convergence in the operator-norm topology for the special family of contractions in a Hilbert space $\mathbf{H}$

$$F(t) = (I + tA)^{-1},$$

where A is an m-sectorial operator and 0 belongs to the resolvent set of A. On the other hand, well-known Chernoff theorem states the convergence in the strong operator topology of a general family of contractions $F(t)$ on a Banach space $\mathbf{B}$. Examining the proof of Theorem 4 one can see that it is possible to formulate sufficient conditions on the family $F(t)$ in order to get the rate of convergence n^{-1} in the Chernoff theorem. Precisely, repeating all steps of the proof of Theorem 4, we have the following result.

Proposition 7. *Let $\{F(t), t \in R_+\}$ be a family of contractions on $\mathbf{B}$ with $F(0) = I$. Suppose that the closure A of the strong derivative $F'(0)$ is the generator of a (C_0) contraction semigroup $T(t) = e^{tA}$. Assume that there exist constants $C_1, \dots, C_4$ depending only on the family F such that, for all $\tau > 0$, $k \geq 1$, and $i = 1, 2$,*

$$\max\Big(\|A^{-i}\big(F(\tau) - T(\tau)\big)\|,\ \|\big(F(\tau) - T(\tau)\big)A^{-i}\|\Big) \leq C_i\tau^i,$$

$$\|A^iT(\tau)\| \leq C_3\tau^{-i}, \quad \|F(\tau)^kA\| \leq \frac{C_4}{\tau k}.$$

Then there exists a constant $\bar{C}$ depending on the family F such that, for all $n \geq 1$,

$$\|\big(F(t/n)\big)^n - T(t)\| \leq \bar{C}n^{-1}.$$

This proposition can be considered as an auxiliary result. It is possible to ask which conditions on F are sufficient to establish the estimates required in Proposition 7. One can guess may be some kind of smoothness of the function $F'(t)$ is a right condition. Another interesting problem is the following one. We know that if, in the CLT, the summands have only moments of the order $2 + \delta$ with $0 < \delta < 1$, then the rate of convergence is of order $n^{-\delta/2}$, and it can be obtained using a smoothing procedure and induction. It is interesting whether it is possible to prove a result similar to Proposition 7 but with the rate $n^{-\alpha}$ with some $0 < \alpha < 1$ instead of n^{-1}.

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