On Beveridge–Nelson decomposition and limit theorems for linear random fields

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Abstract

We consider linear random fields and show how an analogue of the Beveridge–Nelson decomposition can be applied to prove limit theorems for sums of such fields.

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1 Introduction

In a recent paper [15], we considered the self-normalization problem for the particular spatial autoregressive process

$$Y_{t,s} = aY_{t-1,s} + bY_{t,s-1} + \varepsilon_{t,s}, \tag{1}$$

where $\varepsilon_{t,s}$, $(t,s) \in \mathbf{Z}^2$ are i.i.d. random variables with $E\varepsilon_{1,1} = 0$ and $E\varepsilon_{1,1}^2 = 1$, and $|a| + |b| < 1$. Investigating the limit behavior of

$$\frac{\sum_{t,s=1}^{n} Y_{t,s}}{(\sum_{t,s=1}^{n} Y_{t,s}^2)^{1/2}},$$

we essentially used the well-known representation of the stationary process $Y_{t,s}$,

$$Y_{t,s} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j} a^j b^{k-j}\varepsilon_{t-j,s-k+j}.$$

It was clear that the next step in the problem of self-normalization for sums of dependent random fields is to consider general linear fields, as it was done for time series: in [5] self-normalization was considered for a simple $AR(1)$ process, and in [6] for general linear processes.

Let $X_t = \sum_{k=0}^{\infty} c_k\varepsilon_{t-k},\ t \in \mathbf{Z}$, be a linear process, where $\varepsilon_i,\ i \in \mathbf{Z}$, are i.i.d. random variables, and c_i and ε_i are such that X_t is correctly defined (series converges a.s.) and is a stationary process. There are several approaches to investigate sums of (dependent) random variables $\sum_{t=1}^{n} X_t$. One of them is based on the so-called Beveridge–Nelson decomposition (BND) of linear processes, which was successfully used in [6]. This decomposition is a purely algebraic identity and can be easily formulated. Let, as usual, L denote the lag operator ($L\varepsilon_i = \varepsilon_{i-1}$). Then BND can be formulated as follows.

Proposition 1. ([1] or [16]). *Let $C(L) = \sum_{k=0}^{\infty} c_k L^k$. Then*

$$C(L) = C(1) - (1-L)\tilde{C}(L),$$

where $\tilde{C}(L) = \sum_{k=0}^{\infty}\tilde{c}_k L^k,\ \tilde{c}_k = \sum_{j=k+1}^{\infty} c_j$. If $p \geq 1,$ then

$$\sum_{j=1}^{\infty} j^p|c_j|^p < \infty \ \Rightarrow\ \sum_{k=0}^{\infty}|\tilde{c}_k|^p < \infty \quad and \quad |C(1)| < \infty.$$

If $0 < p < 1,$ then

$$\sum_{j=1}^{\infty} j|c_j|^p < \infty \ \Rightarrow\ \sum_{k=0}^{\infty}|\tilde{c}_k|^p < \infty.$$

It is important that this decomposition can be applied to an arbitrary sequence $\{\varepsilon_i\}$; namely, if $X_t = C(L)\varepsilon_t$, then

$$a_n^{-1} \sum_{t=1}^{n} X_t = C(1) a_n^{-1} \sum_{t=1}^{n} \varepsilon_t + R_n, \tag{2}$$

where R_n is of relatively simple structure. Having (2), the next step is to prove that, under appropriate moment conditions on $\{\varepsilon_i\}$ (which usually are assumed to be i.i.d. or martingale differences) and on the coefficients c_i, $R_n \to 0$ in probability or a.s. Thus, limit theorems for $\sum_{t=1}^{n} X_t$ are reduced to the corresponding limit theorems for $\sum_{t=1}^{n} \varepsilon_t$. Using this approach, it is possible to prove the Law of Large Numbers (LLN), Strong LLN (SLLN), Central Limit Theorem (CLT), and Invariance Principle (IP). The existence of variances of ε_t and X_t is not essential, and it is possible to investigate the case where ε_i's are heavy-tailed. It is also possible to use BND when considering the limit behavior of $\sum_{t=1}^{n} X_t^2$ (such sums appear while considering self-normalization, but they are also important in other problems). All these possibilities are demonstrated in the fundamental paper [16] by Phillips and Solo, which was inspiring for this paper.

It was not difficult to write decomposition for a general linear field (see (4) and (12) bellow), and, as application of this decomposition, we proved SLLN and CLT for sums $\sum_{t,s \in D_n} X_{t,s}$, where D_n is some increasing sequence of subsets of $\mathbf{Z}^2$, and $X_{t,s}$ is a linear field. Later we found that such a decomposition was obtained in a recent paper [10] by Marinucci and Poghosyan; on the other hand, working on BND analogue for fields without knowing the results of [10] had some advantage: we proved some new relations (that were absent in [10]) between the initial coefficients of a linear field and the coefficients in the decomposition.

It is possible to formulate results in the case of random fields over $\mathbf{Z}^d$; however, since there is no essential difference for all dimensions $d \geq 2$, except the notation and formulations that are not so transparent, we restricted ourselves mainly to the case $d = 2$, and only in Subsection 2.5.1 we formulate the decomposition for general d. One can say that the situation for LLN and CLT for linear fields is almost the same as that for linear processes, while for SLLN and IP, we noticed some differences: even for the most simple sets, squares, we were able to prove SLLN for random fields under the condition $E|\varepsilon_{t,s}|^{1+\beta} < \infty$, $\beta > 0$. On the other hand, it is clear that, for such simple sets, the result holds for $\beta = 0$ and it is natural to expect that it can be proved using BND. For rectangles, the situation is more complicated, see Subsection 2.5.2. The same situation is for IP. If in [16] IP (the convergence to a Wiener process) was proved under a natural second-moment condition, all our attempts to prove (we tried direct and indirect applications of BND) an analogous result for fields failed; all calculations showed that the existence of moments of order > 4 is needed. The same result is in the above-cited paper [10], where IP for linear random fields (for an arbitrary d) was proved assuming the existence of moments of innovations of order $q > 2d$. Thus, the question whether the IP for linear random fields can be proved using BND and under a second-moment condition remains open.

The paper is organized as follows. In Section 2 we formulate the obtained results. In Subsection 2.1 BND for random fields and some properties of coefficients are formulated. In Subsection 2.2 we directly apply BND to prove SLLN and CLT. In Subsection 2.3, as in [16], we indirectly use BND to obtain new results for SLLN and CLT. In Subsection 2.4 a limit theorem for the sum $\sum_{t,s\in D_n} X_{t,s}^2$ is formulated. In the last Subsection 2.5 various generalizations and some directions of further research are discussed. In Section 3 we collected auxiliary lemmas, and Section 4 contains the proofs of the results stated in Section 2.

2 Formulation of results

2.1 Decomposition of linear random fields

Let

$$X_{t,s} = \sum_{k,l\geq 0} \varphi_{k,l}\varepsilon_{t-k,s-l}, \quad (t,s) \in \mathbf{Z}^2, \tag{3}$$

be a linear random field. We assume that the i.i.d. random variables $\varepsilon_{t,s}$, $(t,s) \in \mathbf{Z}^2$ and coefficients $\varphi_{k,l}$ are such that the series defining $X_{t,s}$ converges a.s. Let $\mathbf{L} = (L_1, L_2)$ be the lag operators defined by

$$L_1\varepsilon_{t,s} = \varepsilon_{t-1,s}, \;\; L_2\varepsilon_{t,s} = \varepsilon_{t,s-1}.$$

We denote by $\mathcal{L}_{q,p}$ the condition

$$\sum_{k,l\geq 0} (k^*l^*)^q |\varphi_{k,l}|^p < \infty,$$

where $i^* = i$ for $i \geq 1$ and $0^* = 1$; we also denote $\mathcal{L}_p := \mathcal{L}_{p,p}$. Let

$$\Phi(\mathbf{L}) = \sum_{k,l\geq 0} \varphi_{k,l} L_1^k L_2^l.$$

To formulate the main result of this subsection, we need the following notation:

$$\mu_1 = \Phi(1,1) = \sum_{k,l\geq 0} \varphi_{k,l},$$

$$A_2(\mathbf{L}) = \Phi^*(\mathbf{L})\Delta_2(\mathbf{L}),\ \Delta_2(\mathbf{L}) = (1-L_1)(1-L_2),$$

$$\Phi^*(\mathbf{L}) = \sum_{k,l\geq 0} \varphi^*_{k,l} L_1^k L_2^l, \quad \varphi^*_{k,l} = \sum_{i\geq k+1, j\geq l+1} \varphi_{i,j},$$

$$A_1(\mathbf{L}) = B(L_1)\Delta_1(L_1) + D(L_2)\Delta_1(L_2), \quad \Delta_1(L_i) = (1-L_i),$$

$$B(L_1) = \sum_{j\geq 0} b_j L_1^j, \quad b_j = \varphi^*_{j,-1} = \sum_{i\geq j+1, k\geq 0} \varphi_{i,k},$$

$$D(L_2) = \sum_{j\geq 0} d_j L_2^j, \quad d_j = \varphi^*_{-1,j} = \sum_{i\geq 0, k\geq j+1} \varphi_{i,k}.$$

.

Theorem 2. *The following identity holds:*

$$\Phi(\mathbf{L}) = \mu_1 + A_2(\mathbf{L}) - A_1(\mathbf{L}). \tag{4}$$

The relations

$$\sum_{k,l\geq 0} |\varphi^*_{k,l}|^p < \infty,\ \sum_{j\geq 0} |b_j|^p < \infty,\ \sum_{j\geq 0} |d_j|^p < \infty,\ \mu_1 < \infty \tag{5}$$

hold if either condition $\mathcal{L}_p$ in the case $1 \leq p < \infty$ or condition $\mathcal{L}_{1,p}$ in the case $0 < p < 1$ is satisfied.

As already mentioned, relation (4) was proved in [10], while the relation for the coefficients is new. Since there is an interplay between the moment conditions for innovations $\varepsilon_{t,s}$, $(t,s) \in \mathbf{Z}^2$, and conditions on the coefficients $\varphi_{k,l}$, such relations are important. In [10] the only condition on the coefficients is formulated as

$$\sum_{k,l\geq 0} \sum_{i\geq k+1, j\geq l+1} |\varphi_{i,j}| < \infty,$$

which is essentially condition $\mathcal{L}_1$ in our notation.

2.2 Direct application of the decomposition

One and probably the main application of BND of linear random fields is limit theorems for appropriately normalized sums $\sum_{t,s\in D_n} X_{t,s}$, where D_n is some increasing sequence of subsets of $\mathbf{Z}^2$, and normalizing constants depend on the cardinality of sets D_n. We shall take the most simple sets $D_n = \{(i,j) \in \mathbf{Z}^2 : 1 \le i \le n,\ 1 \le j \le n\}$. In the last subsection we discuss the possibility to consider more general sets, but in Subsections 2.2–2.4 the notation $\sum_{t,s\in D_n}$ means summation over the above-written square. Let us denote

$$S_n = \sum_{t,s\in D_n} X_{t,s}, \quad Z_n = \sum_{t,s\in D_n} \varepsilon_{t,s}.$$

Remark 3. Since the main goal of the paper is to show the reduction from S_n (sum of dependent random variables) to Z_n (sum of i.i.d. random variables), choosing simple sets D_n has the following advantage. Limit theorems for Z_n can be obtained from classical one-dimensional sequences if we take a map $h : \{(i,j) : i \ge 1,\ j \ge 1\} \to \mathbf{N}$ such that $Z_n = \tilde{Z}_{n^2}$, where

$$\tilde{Z}_n = \sum_{k=1}^{n} \tilde{\varepsilon}_k,$$

and $\tilde{\varepsilon}_k = \varepsilon_{t,s}$ if $k = h(t,s)$.

In the sequel, we shall use this observation without mentioning.

From relation (4) we get the following result.

Proposition 4. *The following relation holds:*

$$S_n = \mu_1 Z_n + R_n, \tag{6}$$

where

$$\begin{aligned} R_n &= \xi_{n,n} - \xi_{n,0} - \xi_{0,n} + \xi_{0,0} \\ &+ \eta_{n,n} - \eta_{0,n} + \zeta_{n,n} - \zeta_{n,0}, \end{aligned} \tag{7}$$

$$\xi_{t,s} = \Phi^*(\mathbf{L})\varepsilon_{t,s} = \sum_{k,l\ge 0} \varphi^*_{k,l}\varepsilon_{t-k,s-l},$$

$$\eta_{t,n} = \sum_{s=1}^{n} \bar{\varepsilon}_{t,s}, \quad \bar{\varepsilon}_{t,s} = B(L_1)\varepsilon_{t,s} = \sum_{j\ge 0} b_j\varepsilon_{t-j,s},$$

$$\zeta_{n,s} = \sum_{t=1}^{n} \hat{\varepsilon}_{t,s}, \quad \hat{\varepsilon}_{t,s} = D(L_2)\varepsilon_{t,s} = \sum_{j\ge 0} d_j\varepsilon_{t,s-j}.$$

From this proposition it is clear that limit theorems for S_n are reduced to limit theorems for sums of i.i.d. random variables if we prove that, after appropriate normalization, the remainder term R_n tends to zero (in probability or a.s.). By direct application of BND, as in [16], we mean that for estimation of R_n, we use (7), while indirect application means the estimation of R_n using the equality

$$R_n = S_n - \mu_1 Z_n,$$

that is, only the fact that S_n is approximated by $\mu_1 Z_n$. We say that SLLN and CLT hold for S_n if

$$n^{-2} S_n \xrightarrow{a.s.} 0$$

and

$$n^{-1} S_n \xrightarrow{d} N(0, \mu_1^2),$$

respectively. Here $N(0, \sigma^2)$ stands for a normal random variable with mean zero and variance σ^2 (and, as is usual in limit theorems, the same notation is used for the distribution of this normal random variable).

The first and rather easily obtained result can be formulated as follows.

Theorem 5. *Suppose that $\varepsilon_{t,s},\ (t,s) \in \mathbf{Z}^2$ are i.i.d. random variables with $E\varepsilon_{00} = 0,\ E\varepsilon_{00}^2 = 1$, condition $\mathcal{L}_2$ holds, and $\mu_1 \neq 0$. Then SLLN and CLT for S_n hold.*

The existence of the second moment of $\varepsilon_{0,0}$ is natural for CLT to hold (most probably, it is possible to prove CLT for S_n with appropriate normalization using BND under the assumption that $\varepsilon_{0,0}$ belongs to the domain of the attraction of a normal law), but this is not the case for SLLN. The assumptions $E\varepsilon_{00} = 0$ and $E|\varepsilon_{00}| < \infty$, together with some condition on coefficients $\{\varphi_{k,l}\}$, would be natural for SLLN to hold. Such a result for linear processes is obtained in [16]. Our result is a little bit weaker.

Theorem 6. *Suppose that $\varepsilon_{t,s},\ (t,s) \in \mathbf{Z}^2$ are i.i.d. random variables with $E\varepsilon_{00} = 0$ and $E|\varepsilon_{00}|^{1+\beta} < \infty$ and that condition $\mathcal{L}_{1+\beta}$ holds for some $1 \geq \beta > 0$. Then SLLN for S_n holds.*

By direct approach it is not difficult to prove a limit theorem in the case where innovations have infinite variance. Let us assume that $\varepsilon_{0,0}$ belongs to the normal domain of attraction of a stable random variable η_α with $0 < \alpha < 2,\ \ E\varepsilon_{00} = 0$, if $\alpha > 1$, and $\varepsilon_{0,0}$ is a symmetric random variable if $\alpha = 1$. The normal domain of attraction is assumed only for simplicity, in order to avoid slowly varying functions in formulations. Such assumptions mean that

$$n^{-2/\alpha} Z_n \xrightarrow{d} \eta_\alpha.$$

Theorem 7. *If $\varepsilon_{0,0}$ satisfies the above-formulated assumptions, condition $\mathcal{L}_{1,\alpha}$ if $0 < \alpha < 1$ or $\mathcal{L}_\alpha$ if $1 \le \alpha < 2$ holds, and $\mu_1 \neq 0$, then*

$$n^{-2/\alpha} S_n \xrightarrow{d} \mu_1 \eta_\alpha.$$

In [11] such a limit result for rectangles instead of squares and under a weaker assumption on the coefficients $\varphi_{k,l}$ is proved in the case $1 < \alpha < 2$, but the proof is much more involved.

2.3 Indirect application of the decomposition

In this subsection from the decomposition we use only the fact that S_n is approximated by $\mu_1 Z_n$. We have the following result.

Theorem 8. *Suppose that $\varepsilon_{t,s}$, $(t,s) \in \mathbf{Z}^2$, are i.i.d. random variables with $E\varepsilon_{00} = 0$, and conditions $\sum_{k,l\ge 0} |\varphi_{k,l}| < \infty$ and $\mu_1 \neq 0$ hold. If $E|\varepsilon_{00}|^2 = 1$, then CLT for S_n holds, and if $E|\varepsilon_{00}|^{1+\beta} < \infty$ for some $1/2 < \beta \le 1$, then SLLN for S_n holds.*

2.4 Limit theorem for sums of squares

In [16] it is shown that BND can be useful to prove limit results for $\sum_t X_t^2$, where $\{X_t,\ t \in \mathbf{Z}\}$ is a linear process. Similarly, decomposition (4) can be used to investigate limit properties of $\sum_{D_n} X_{t,s}^2$ with $X_{t,s}$ being a linear random field. Although we must admit that the notation and proofs become more complicated, on the other hand, it is difficult to believe that there can be a very simple approach to deal with such sums. To formulate our result, we need some more notation. We set (always keeping in mind that $\varphi_{k,l} = 0$ if $k < 0$ or $l < 0$)

$$\psi_{k,l,\pm p,\pm r} = \varphi_{k,l}\varphi_{k\pm p,l\pm r}, \quad k,l,p,r \ge 0,$$

$$\psi^*_{k,l,\pm p,\pm r} = \sum_{\substack{i\ge k+1 \\ j\ge l+1}} \psi_{i,j,\pm p,\pm r}, \quad k,l,p,r \ge 0,$$

$$\Psi_{\pm p,\pm r}(\mathbf{L}) = \sum_{k,l\ge 0} \psi_{k,l,\pm p,\pm r} L_1^k L_2^l, \quad \Psi^*_{\pm p,\pm r}(\mathbf{L}) = \sum_{k,l\ge 0} \psi^*_{k,l,\pm p,\pm r} L_1^k L_2^l,$$

$$b_{j,\pm p,\pm r} = \psi^*_{j,-1,\pm p,\pm r} = \sum_{\substack{k\ge 0 \\ i\ge j+1}} \varphi_{i,k}\varphi_{i\pm p,k\pm r}, \ j,p,r \ge 0,$$

$$d_{j,\pm p,\pm r} = \psi^*_{-1,j,\pm p,\pm r} = \sum_{\substack{k\geq 0\\ i\geq j+1}} \varphi_{k,i}\varphi_{k\pm p,i\pm r},\ \ j,p,r \geq 0.$$

We also define that lag operators act for product of random variables in the following way:

$$L_1^k L_2^l \varepsilon_{t,s}\eta_{u,v} = \varepsilon_{t-k,s-l}\eta_{u-k,v-l}.$$

Proposition 9. a) *The following formal relation holds:*

$$\begin{aligned} X_{t,s}^2 &= \mu_2\varepsilon_{t,s}^2 + (A_{0,0}(\mathbf{L}) - B_{0,0}(L_1) - C_{0,0}(L_2))\varepsilon_{t,s}^2 \\ &+ \textstyle\sum^*_{p,r\geq 1}(\mu_{\pm p,\pm r} + A_{\pm p,\pm r}(\mathbf{L}) - B_{\pm p,\pm r}(L_1) - C_{\pm p,\pm r}(L_2))\varepsilon_{t,s}\varepsilon_{t\mp p,s\mp r}, \end{aligned} \tag{8}$$

where $\sum^*$ *means that there are four sums with all possible combinations of signs, and, for all* $p, r \geq 0$*,*

$$\mu_2 = \sum_{k,l\geq 0} \psi_{k,l}^2,$$

$$\mu_{\pm p,\pm r} = \Psi_{\pm p,\pm r}(1,1) = \sum_{k,l\geq 0} \psi_{k,l,\pm p,\pm r},$$

$$A_{\pm p,\pm r}(\mathbf{L}) = \Psi^*_{\pm p,\pm r}(\mathbf{L})\Delta_2(\mathbf{L}) = \sum_{k,l\geq 0} \psi^*_{k,l,\pm p,\pm r} L_1^k L_2^l (1-L_1)(1-L_2),$$

$$B_{\pm p,\pm r}(L_1) = \sum_{j\geq 0} b_{j,\pm p,\pm r} L_1^j (1-L_1)$$

$$C_{\pm p,\pm r}(L_2) = \sum_{j\geq 0} d_{j,\pm p,\pm r} L_2^j (1-L_2).$$

b) *Summing relations (8), we get*

$$\sum_{D_n} X_{t,s}^2 = \mu_2 \sum_{D_n} \varepsilon_{t,s}^2 + R_{n,2}, \tag{9}$$

where $R_{n,2}$ *is obtained in obvious way, and its expression will be given in the proof of the proposition.*

The properties of the coefficients involved in (8) will be given in Lemma 15.

Relation (9) means that SLLN or CLT for squares of $X_{t,s}$ is reduced for SLLN or CLT for squares of $\varepsilon_{t,s}$ if we are able to prove that after appropriate normalization the remainder term in (9) tends to zero a.s. or in probability.

As an example of the result obtained in this way, we formulate the following theorem. Without loss of generality we may assume that, for all $k, l \geq 0$, $\varphi_{k,l} \neq 0$. We shall require the following technical condition: for some $\bar{C} \geq 1$ and all $k, l \geq 0$,

$$\varphi^2_{i+k,j+l} \leq \bar{C}\varphi^2_{i,j}. \tag{10}$$

Theorem 10. *Assume that* $E\varepsilon_{0,0} = 0$, $E\varepsilon^2_{0,0} = 1$, *and* $\mathcal{L}_{1+\varepsilon,2}$ *for some* $\varepsilon > 0$ *and (10) hold. Then*

$$n^{-2}\sum_{D_n} X^2_{t,s} \xrightarrow{a.s.} \mu_2 \;= EX^2_{0,0} = \sum_{k,l\geq 0} \varphi^2_{k,l}.$$

2.5 Various generalizations

In this section we present several possible generalizations of the results stated in the previous subsections.

2.5.1 Decomposition in higher dimensions

As was mentioned, BND for general d-dimensional linear fields was written in [10], but it seems that, for practical work with this decomposition, it is better to write it in a little bit different form, separating terms having the same "degree" of differencing operator $1 - L_i$, as this was done in (4). To make this statement more clear, we shall write this form of decomposition in the case $d = 3$. Let now $\mathbf{L} = (L_1, L_2, L_3)$, let $\mathbf{L}_j$ be obtained from $\mathbf{L}$ by dropping L_j from the latter, and let $\Delta_3(\mathbf{L}) = (1 - L_1)(1 - L_2)(1 - L_3)$. Denote

$$\Phi(\mathbf{L}) = \sum_{k,l,m\geq 0} \varphi_{k,l,m} L_1^k L_2^l L_3^m.$$

Then we have the formal decomposition

$$\Phi(\mathbf{L}) = \Phi(\mathbf{1}) - A_3(\mathbf{L}) + A_2(\mathbf{L})) - A_1(\mathbf{L}), \tag{11}$$

where

$$A_3(\mathbf{L}) = \Phi^*(\mathbf{L})\Delta_3(\mathbf{L}),$$

$$A_2(\mathbf{L}) = \sum_{j=1}^{3} \Phi^*_{2,j}(\mathbf{L}_j)\Delta_2(\mathbf{L}_j), \quad A_1(\mathbf{L}) = \sum_{j=1}^{3} \Phi^*_{1,j}(L_j)\Delta_1(L_j),$$

$$\Phi^*(\mathbf{L}) = \sum_{k,l,m\geq 0} \varphi^*_{k,l,m} L_1^k L_2^l L_3^m, \quad \varphi^*_{k,l,m} = \sum_{i\geq k+1, j\geq l+1, h\geq m+1} \varphi_{i,j,h},$$

and it is clear how to write expressions for $\Phi^*_{2,j}(\mathbf{L}_j)$ and $\Phi^*_{1,j}(L_j)$. For arbitrary d, BND has the form

$$\Phi(\mathbf{L}) = \Phi(\mathbf{1}) + \sum_{i=1}^{d} (-1)^i A_i(\mathbf{L}), \tag{12}$$

where now $\mathbf{L} = (L_1, \ldots, L_d)$, $\mathbf{1} = (1, \ldots, 1)$, with obvious definitions of polynomials $A_i(\mathbf{L})$. The relations between coefficients in the general case also can be easily obtained.

Having the decomposition (11), one can obtain limit theorems for sums $\sum_{D_n} X_{t,s,u}$, where now $D_n = [1, n]^3 \cap \mathbf{Z}^3$ (the same can be said about the case of arbitrary d). To this aim, one can write

$$\sum_{D_n} X_{t,s,u} = \Phi(\mathbf{1}) \sum_{D_n} \varepsilon_{t,s,u} + \bar{R}_n,$$

where $\bar{R}_n = R_{n,1} + R_{n,2} + R_{n,3}$ and $R_{n,i} = \sum_{D_n} A_i(\mathbf{L})\varepsilon_{t,s,u}$. Each term of the remainder $\bar{R}_n$ has a rather simple structure (the same as that in the case $d = 2$); for example, $R_{n,3} = \sum_{D_n} A_3(\mathbf{L})\varepsilon_{t,s,u}$ can be written as

$$\hat{\varepsilon}_{n,n,n} - \hat{\varepsilon}_{n,n,0} - \hat{\varepsilon}_{n,0,n} - \hat{\varepsilon}_{0,n,n} + \hat{\varepsilon}_{n,0,0} + \hat{\varepsilon}_{0,n,0} + \hat{\varepsilon}_{0,0,n} - \hat{\varepsilon}_{0,0,0},$$

where

$$\hat{\varepsilon}_{t,s,u} = \Phi^*(\mathbf{L})\varepsilon_{t,s,u} = \sum_{k,l,m \geq 0} \varphi^*_{k,l,m} \varepsilon_{t-k,s-l,u-m}.$$

The terms $R_{n,1}$ and $R_{n,2}$ can be written similarly. Having these expressions of $\bar{R}_n$, it is not difficult (under appropriate conditions) to generalize the results of Subsections 2.2 and 2.3 (we did not consider $\sum_{D_n} X^2_{t,s,u}$ only for the reason that calculations became too lengthy).

2.5.2 More general regions

Now we return to the case $d = 2$ and discuss the possibility to consider more general regions than squares $D_n = [1, n]^2 \cap \mathbf{Z}^2$. As was demonstrated, the success of the decomposition (both in the cases $d = 1$ and $d \geq 2$) depends on the fact that the structure of the remainder R_n obtained after summing over D_n is rather simple, and this fact is due to the simplicity of the boundary of D_n (in the case $d = 1$, the set D_n is simply an interval, and the boundary consists of two points). It is clear that if we take an arbitrary increasing sequence of sets $A_n \subset \mathbf{Z}^2$, the structure of the remainder term R_n will be too complicated, and application of BND (at least, the direct approach) will be useless. On the other hand, if instead of the squares $D_n = [1,n]^2 \cap \mathbf{Z}^2$ we take the rectangles $D_{\mathbf{n}} = [1,n_1] \times [1,n_2] \cap \mathbf{Z}^2$, where $\mathbf{n} = (n_1,n_2)$, the structure of the remainder term remains essentially the same. For example, denoting

$$S_{\mathbf{n}} = \sum_{t,s \in D_{\mathbf{n}}} X_{t,s}, \quad Z_{\mathbf{n}} = \sum_{t,s \in D_{\mathbf{n}}} \varepsilon_{t,s},$$

(6) now can be written as follows:

$$S_{\mathbf{n}} = \mu_1 Z_{\mathbf{n}} + R_{\mathbf{n}}, \tag{13}$$

where

$$\begin{aligned} R_{\mathbf{n}} &= \xi_{n_1,n_2} - \xi_{n_1,0} - \xi_{0,n_2} + \xi_{0,0} \\ &+ \eta_{n_1,n} - \eta_{0,n_2} + \zeta_{n_1,n_2} - \zeta_{n_1,0}. \end{aligned} \tag{14}$$

Having (14), it is not difficult to see that one can restate CLT or state LLN for $S_{\mathbf{n}}$ assuming that $\min(n_1,n_2) \to \infty$ under the same moment conditions and conditions on $\{\varphi_{k,l}\}$ as it was done for S_n. We recall that useful Lemma 13 is formulated for general rectangles. However, the situation is different for SLLN, and passing from S_n to $S_{\mathbf{n}}$ is not trivial. As an example, let us take Theorem 5. Instead of (31), now we must prove that

$$(n_1 n_2)^{-1} R_{\mathbf{n}} \xrightarrow{a.s.} 0$$

as $\min(n_1,n_2) \to \infty$, and this will follow if we prove

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} P(|(n_1 n_2)^{-1} R_{\mathbf{n}}| > \epsilon) \le \infty.$$

There is no problem for terms with $\xi_{n_1,n_2},\ \xi_{0,n_2},\ \xi_{n_1,0}$ (see (14)), but for the terms with η or ζ, using moment inequalities, we have

$$P(|(n_1 n_2)^{-1} \Big| \sum_{s=1}^{n_2} \bar{\varepsilon}_{n_1,s} \Big| > \epsilon) \le \epsilon^{-2} (n_1 n_2)^{-2} \sum_{s=1}^{n_2} E \bar{\varepsilon}^2_{n_1,s} \le C \epsilon^{-2} n_1^{-2} n_2^{-1},$$

and with respect to n_2, we get the divergent harmonic series. To save the situation, one can require the stronger moment condition $E|\varepsilon_{1,1}|^{2+\delta} < \infty$ or to use a little bit stronger normalization, that is, $(n_1 n_2)^{-1} (\ln n_1 \ln n_2)^{-\gamma}$ with $\gamma > 1/2$. A similar situation is with another result on SLLN, and even SLLN for $Z_{\mathbf{n}}$ is different, comparing with Z_n: it is known (see [18]) that

$$(n_1 n_2)^{-1} Z_{\mathbf{n}} \xrightarrow{a.s.} 0$$

iff $E|\varepsilon_{1,1}| \ln |\varepsilon_{1,1}| < \infty$ (here $\ln t = 1$ for $0 < t < e$) and $E\varepsilon_{1,1} = 0$.

Between squares and general rectangles, there are intermediate possibilities when we consider rectangles indexed by n and defined by means of the relations $n_1 = n, n_2 = g(n)$, where g is some integer-valued function. The case $g(n) = n$ gives us squares (here it is worth to mention that the remark before Proposition 4 can be applied in the case of the increasing function g). In other words, it is possible to consider the convergence of double-indexed sequence $S_{\mathbf{n}}$ along some path. In [17] there is developed some general theory of the so-called sequential and joint convergence of double-indexed processes having a specific structure, connected with panel data in econometrics.

We intend to investigate SLLN for rectangles and even more general sets A_n which are convex and have "small" boundary compared with interior points in the nearest future.

2.5.3 Other possible directions of investigation

As was mentioned in the introduction, the motivation to look at decomposition of linear fields was the problem of self-normalization, since BND of linear processes was successfully applied for this problem in [6]. Combining results on CLT for S_n and LLN for sums of $X_{t,s}^2$ (although we had formulated only SLLN for such sums, it is much easier to obtain LLN), we can get some simple result on self-normalization. However, there are still open problems concerning self-normalization in this context, and they are left for future research.

It was shown in [16] that BND for linear processes, which is purely an algebraic identity and can be applied for any sequence of random variables ε_i's, is a useful tool not only in the case of the i.i.d. sequence but also for martingale-difference sequence. Since martingales on the plane are defined in a specific way (see, for example, [3], [19]) and have some connection with the operator $\Delta_2(\mathbf{L})$, it would be interesting to explore if BND can give similar results for linear fields generated by martingales on the plane.

It would be interesting to look if two-dimensional BND can be applied to asymptotic analysis of panel data. In [17] linear random processes reflecting panel data are considered in a very general setting (in our notation, $\varepsilon_{i,t}$ and $X_{i,t}$ are m-dimensional vectors, and $\{\varphi_{i,t}\}$ are $m \times m$ random matrices), on the other hand, essentially one-dimensional linear processes of the form $X_{i,t} = \sum_{s=1}^{\infty} \varphi_{i,s}\varepsilon_{i,t-s}$ are considered, and usual BND for each i is applied, since it is assumed that the matrices $\{\varphi_{i,s}\}$ are i.i.d. with respect to i.

One more promising direction of investigation is limit theorems for linear random processes and fields with values in a separable Banach space using BND. As an example of a result in this direction, we can provide the following one. Let us consider linear random field (3), assuming now that $\varepsilon_{t,s},\ (t,s) \in \mathbf{Z}^2$ are i.i.d. random elements with values in a separable Banach space B of type 2, $E\varepsilon_{00} = 0,\ E||\varepsilon_{00}||^2 < \infty$, and $\varphi_{k,l}$ are linear bounded operators in B such that

$$\sum_{k,l\geq 0} ||\varphi_{k,l}||_B < \infty$$

and $\mu_1 = \sum_{k,l\geq 0} \varphi_{k,l} \neq \mathbf{0}$. Here $||\cdot||$ and $||\cdot||_B$ denote the norms in B and $L(B)$ (the Banach space of linear bounded operators in B), respectively, and $\mathbf{0}$ stands for the null operator. For probability notions in Banach spaces (types and cotypes, covariance operators, CLT, etc.), we refer to monographs [14] or [9]. Taking into account the previous subsection, we have the following result.

Theorem 11. *Under the above-formulated conditions, the CLT for $S_{\mathbf{n}}$ holds, that is,*

$$\frac{1}{\sqrt{n_1 n_2}} S_{\mathbf{n}} \xrightarrow{d} N(0, \mu_1 A \mu_1^*)$$

as $\min(n_1, n_2) \to \infty$. Here $N(0, C)$ denotes a Gaussian mean zero B-valued random element with covariance operator C, A denotes the covariance operator of $\varepsilon_{1,1}$, and μ_1^* is the adjoint operator of μ_1.

This result can be considered as a small generalization of Theorem 2 of [13], where CLT was proved in the case of separable Hilbert space, and in case $d = 1$ (linear processes) it coincides with Theorem 7.8 of [2] (by the way, formulated without proof and without any reference).

In ending this section one should also mention why we do not compare our results with those previously obtained. The main goal of the paper, as was mentioned earlier, was demonstration that BND is useful in obtaining the results with very simple and short proofs and that this decomposition has a wide area of applications. Limit theorems for random fields (and particularly, for linear ones) are investigated more than 30 years (see, for example, [7], [8], [4]), but most papers in this field are exploiting some mixing or weak-dependence properties, and usually it is not easy to apply such results for linear random fields. One maybe should mention also some disadvantages of BND. Direct application of BND usually does not allow one to obtain optimal results, since passing from $X_{t,s}$ to a new linear random field $\xi_{t,s}$ requires conditions stronger than needed on the coefficients $\varphi_{k,l}$. Even BND itself requires the finiteness of μ_1, while, for the existence of

$X_{t,s}$ in the case of innovations with second moment, the weaker condition $\sum_{k,l\geq 0}\varphi_{k,l}^2 < \infty$ is sufficient. Also one must note that to prove SLLN for sums over comparatively simple sets is much more easy using ergodic theorems. This remark can be addressed to linear processes as well - it is very easy to prove Theorem 3.7 from [16] using ergodic theory.

3 Auxiliary lemmas

We recall that $\Phi(\mathbf{L}) = \sum_{k,l\geq 0}\varphi_{k,l}L_1^kL_2^l$ and $\Phi^*(\mathbf{L}) = \sum_{k,l\geq 0}\varphi^*_{k,l}L_1^kL_2^l$, where $\varphi^*_{k,l} = \sum_{i\geq k+1,j\geq l+1}\varphi_{i,j}$.

Lemma 12. *If condition $\mathcal{L}_p$ for $p\geq 1$ or condition $\mathcal{L}_{1,p}$ for $0<p<1$ is satisfied, then*

$$\sum_{k,l\geq 0}|\varphi^*_{k,l}|^p<\infty,\quad \sum_{j\geq 0}|b_j|^p<\infty,\quad \sum_{j\geq 0}|d_j|^p<\infty,\quad \sum_{k,l\geq 0}\varphi_{k,l}<\infty.$$

Proof. Since $b_j=\varphi^*_{j,-1}$ and $d_j=\varphi^*_{-1,j}$, we shall prove the first relation only for larger summation area. The case $p=1$ is trivial; therefore, we consider $p>1$. Applying Holder's inequality with some a satisfying the inequalities

$$\frac{1}{q}<a<\frac{1}{q}+\frac{1}{p}=1,$$

we have

$$\begin{aligned}
\sum_{k,l\geq -1}\Big|\sum_{\substack{i\geq k+1\\ j\geq l+1}}\varphi_{i,j}\Big|^p &= \sum_{k,l\geq -1}\Big|\sum_{\substack{i\geq k+1\\ j\geq l+1}}(i^*j^*)^a\varphi_{i,j}(i^*j^*)^{-a}\Big|^p\\
&\leq \sum_{k,l\geq -1}\Big(\sum_{\substack{i\geq k+1\\ j\geq l+1}}(i^*j^*)^{ap}|\varphi_{i,j}|^p\Big)\Big(\sum_{\substack{i\geq k+1\\ j\geq l+1}}(i^*j^*)^{-aq}\Big)^{p/q}\\
&\leq C\sum_{k,l\geq -1}\sum_{\substack{i\geq k+1\\ j\geq l+1}}(i^*j^*)^{ap}|\varphi_{i,j}|^p(k^*l^*)^{(1-aq)p/q}\\
&\leq C\sum_{i,j\geq 0}(i^*j^*)^{ap}|\varphi_{i,j}|^p\sum_{k=-1}^{i}k^{*(1-aq)p/q}\sum_{l=-1}^{j}l^{*(1-aq)p/q}\\
&\leq C\sum_{i,j\geq 0}(i^*j^*)^{ap+1+(1-aq)p/q}|\varphi_{i,j}|^p\\
&\leq C\sum_{i,j\geq 0}(i^*j^*|\varphi_{i,j}|)^p,
\end{aligned}$$

where C is a constant, depending on p and not necessarily the same at different places, and $(-1)^* = 1$.

If $0 < p < 1$, then

$$\sum_{k,l\geq -1} |\varphi^*_{k,l}|^p \leq \sum_{k,l\geq -1} \sum_{\substack{i\geq k+1\\ j\geq l+1}} |\varphi_{i,j}|^p \leq \sum_{i,j\geq 0} i^* j^* |\varphi_{i,j}|^p.$$

To see that condition $\mathcal{L}_p$, $p > 1$, implies $\sum_{k,l\geq 0} |\varphi_{k,l}| < \infty$, it suffices to write

$$\sum_{k,l\geq 0} |\varphi_{k,l}| = |\varphi_{0,0}| + \sum_{k\geq 1} |\varphi_{k,0}| + \sum_{l\geq 1} |\varphi_{0,l}| + \sum_{k,l\geq 1} |\varphi_{k,l}|$$

and to apply Holder's inequalities for three last terms of this equality. In the case $p < 1$, without loss of generality, we may assume that all coefficients $|\varphi_{k,l}|$ are less than 1, and the same equality trivially gives the result. Here we see why it is convenient to use the notation i^* and j^* in the conditions. The lemma is proved.

The next lemma is a slight generalization of a similar lemma from [13]. Since the main objects of our paper are real-valued fields with $d = 2$, we formulate our lemma in this setting only for the reason to keep the same notation. Generalization to the case $d > 2$ and for random fields with values in a Banach space is trivial (instead of absolute value one needs to use norm). The set of squares D_n (mainly used in our paper) has to be replaced by two-dimensional rectangles, since there is an essential difference between these two cases in the context of the lemma. We shall use vector notations: $\mathbf{j} = (j_1, j_2)$, $\mathbf{n} = (n_1, n_2)$, $|\mathbf{n}| = n_1 n_2$, $\mathbf{1} = (1, 1)$, $[-\mathbf{x}, \mathbf{x}] = [-x_1, x_1] \times [-x_2, x_2]$, similarly for the open rectangle. For vectors the operations of multiplication, division, inequalities, taking integer part of a vector are coordinate-wise, for example, $[\mathbf{nx}] = ([n_1 x_1], [n_2 x_2])$. The use of this notation also indicates that generalization to higher dimensions is not difficult. In our lemma, $\mathbf{n} \to \infty$ means that $\min(n_1, n_2) \to \infty$.

Lemma 13. *Let $\{b_{\mathbf{j}},\ \mathbf{j} \in \mathbf{Z}^2\}$ be real numbers such that*

$$\sum_{\mathbf{j}\in\mathbf{Z}^2} |b_{\mathbf{j}}| < \infty \tag{15}$$

and

$$\sum_{\mathbf{j}\in\mathbf{Z}^2} b_{\mathbf{j}} = 0. \tag{16}$$

Then, for $1 < p \le 2$,

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\in\mathbf{Z}^2}\Big|\sum_{\mathbf{1}-\mathbf{j}\le\mathbf{i}\le\mathbf{n}-\mathbf{j}} b_{\mathbf{i}}\Big|^p \to 0 \qquad \text{as } \mathbf{n}\to\infty. \tag{17}$$

Remark 14. In [13] this lemma (for elements from normed space) was proved for $p = 2$ under the weaker assumption that $|\mathbf{n}| \to \infty$ and the coordinates of $\mathbf{n}$ are nondecreasing. This allows the situation $n_1 \to \infty$, n_2 is constant, while in our formulation both coordinates must tend to infinity. Although we believe that the result of Theorem 2 from [13] remains true under this weaker assumption on the growth of $\mathbf{n}$ (this also defines the growth of summation region $D_{\mathbf{n}}$), the proof of Lemma 1 from [13] contains several mistakes (which we shall point out in our proof), and we do not know how to prove this lemma under this weaker assumption on the growth of $\mathbf{n}$.

Proof. Let us denote

$$A_{\mathbf{n}} = \sum_{\mathbf{j}\notin(-\mathbf{n},\mathbf{n})} |b_{\mathbf{j}}|.$$

Then from (15) it follows that $A_{\mathbf{n}} \to 0$ as $\mathbf{n} \to \infty$. (Here is the first mistake: $A_{\mathbf{n}}$ does not need to tend to zero if one coordinate of $\mathbf{n}$ (at least one coordinate in the case of arbitrary $d > 2$) remains fixed.) Again using (15), it is easy to get

$$\begin{aligned}
&\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\notin(-2\mathbf{n},2\mathbf{n})}\Big|\sum_{\mathbf{1}-\mathbf{j}\le\mathbf{i}\le\mathbf{n}-\mathbf{j}} b_{\mathbf{i}}\Big|^p \le \frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\notin(-2\mathbf{n},2\mathbf{n})}\Big(\sum_{\mathbf{1}-\mathbf{j}\le\mathbf{i}\le\mathbf{n}-\mathbf{j}} |b_{\mathbf{i}}|\Big)^p \\
\le\;& \frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\notin(-2\mathbf{n},2\mathbf{n})}\Big(\sum_{\mathbf{1}-\mathbf{j}\le\mathbf{i}\le\mathbf{n}-\mathbf{j}} |b_{\mathbf{i}}|\Big)\Big(\sum_{\mathbf{i}\notin(-\mathbf{n},\mathbf{n})} |b_{\mathbf{i}}|\Big)^{p-1} \\
\le\;& A_{\mathbf{n}}^{p-1}\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\in\mathbf{Z}^2}\Big(\sum_{\mathbf{1}-\mathbf{j}\le\mathbf{i}\le\mathbf{n}-\mathbf{j}} |b_{\mathbf{i}}|\Big) = A_{\mathbf{n}}^{p-1}\sum_{\mathbf{j}\in\mathbf{Z}^2}|b_{\mathbf{j}}| \to 0 \text{ as } \mathbf{n}\to\infty.
\end{aligned} \tag{18}$$

It remains to prove that

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\in(-2\mathbf{n},2\mathbf{n})}\Big|\sum_{\mathbf{1}-\mathbf{j}\le\mathbf{i}\le\mathbf{n}-\mathbf{j}} b_{\mathbf{i}}\Big|^p \to 0 \text{ as } \mathbf{n}\to\infty. \tag{19}$$

We introduce the function

$$h_{\mathbf{n}}(\mathbf{x}) = \Big|\sum_{\mathbf{1}-[\mathbf{n}\mathbf{x}]\le\mathbf{i}\le\mathbf{n}-[\mathbf{n}\mathbf{x}]} b_{\mathbf{i}}\Big|^p$$

for all $\mathbf{x} \in [-\mathbf{2}, \mathbf{2}]$. From (15) and (16) we have that, for all $\mathbf{x}$ such that $x_i \neq 0, \pm 1$, $h_{\mathbf{n}}(\mathbf{x}) \to 0$ as $\mathbf{n} \to \infty$ (here it is essential that all coordinates of $\mathbf{n}$ tend to infinity) and

$$|h_{\mathbf{n}}(\mathbf{x})| \leq \Big(\sum_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}} |b_{\mathbf{i}-[\mathbf{n}\mathbf{x}]}| \Big)^p \leq \Big(\sum_{\mathbf{i} \in \mathbf{Z}^2} |b_{\mathbf{i}}| \Big)^p < \infty.$$

Having these properties of $h_{\mathbf{n}}(\mathbf{x})$ and using the Lebesgue dominated convergence theorem, it is not difficult to get (19):

$$\begin{aligned} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{j} \in (-\mathbf{2n}, \mathbf{2n}-\mathbf{1})} \Big| \sum_{\mathbf{1}-\mathbf{j} \leq \mathbf{i} \leq \mathbf{n}-\mathbf{j}} b_{\mathbf{i}} \Big|^p &= \frac{1}{|\mathbf{n}|} \sum_{\mathbf{j} \in (-\mathbf{2n}, \mathbf{2n}-\mathbf{1})} h_{\mathbf{n}}(\mathbf{j}/\mathbf{n}) \\ &= \sum_{\mathbf{j} \in (-\mathbf{2n}, \mathbf{2n}-\mathbf{1})} \int_{[\mathbf{j}/\mathbf{n}, (\mathbf{j}+\mathbf{1})/\mathbf{n}]} h_{\mathbf{n}}(\mathbf{x}) d\mathbf{x} \\ &= \int_{[-\mathbf{2}, \mathbf{2}]} h_{\mathbf{n}}(\mathbf{x}) d\mathbf{x} \to 0 \text{ as } \mathbf{n} \to \infty. \end{aligned}$$

In the proof of this lemma in [13], the definition of $h_{\mathbf{n}}(\mathbf{x})$ is slightly different, summation is over the region $\mathbf{1} - [|\mathbf{n}|\mathbf{x}] \leq \mathbf{i} \leq \mathbf{n} - [|\mathbf{n}|\mathbf{x}]$, and this allows one to use the weaker assumption on the growth of $\mathbf{n}$, but then there is a mistake in the equality

$$\int_{[\mathbf{j}/|\mathbf{n}|, (\mathbf{j}+\mathbf{1})/|\mathbf{n}|]} d\mathbf{x} = \frac{1}{|\mathbf{n}|},$$

since, in fact, this last integral is equal to $\frac{1}{|\mathbf{n}|^d}$. From (18) and (19) we get (17), and the lemma is proved.

In the following lemma, we collected properties of various coefficients present in Proposition 9.

Lemma 15. *If conditions $\mathcal{L}_{1+\varepsilon,2}$ for some $\varepsilon > 0$ and (10) are satisfied, then we have that the relations*

$$\sum_{k,l \geq 0} \sum_{p,r \geq 1} (\psi^*_{k,l,\pm p,\pm r})^2 < \infty, \quad \sum_{j \geq 0} \sum_{p,r \geq 1} b^2_{j,\pm p,\pm r} < \infty, \quad \sum_{j \geq 0} \sum_{p,r \geq 1} d^2_{j,\pm p,\pm r} < \infty \tag{20}$$

hold for all combinations of signs, and

$$\sum_{k,l \geq 0} (\psi^*_{k,l,0,0})^2 < \infty, \quad \sum_{j \geq 0} b^2_{j,0,0} < \infty, \quad \sum_{j \geq 0} d^2_{j,0,0} < \infty. \tag{21}$$

If condition $\mathcal{L}_{1/2,2}$ is satisfied, then the relations

$$\sum_{p,r\geq 1} \mu^2_{\pm p,\pm r} < \infty \tag{22}$$

also hold for all combinations of signs.

Proof. Again, as in the proof of lemma 12, it suffices to prove the first relation in (20), since $b_{j,\pm p,\pm r} = \psi^*_{j,-1,\pm p,\pm r}$ and $d_{j,\pm p,\pm r} = \psi^*_{-1,j,\pm p,\pm r}$, therefore, as in the proof of lemma 12, we consider larger area of summation. We start with one combination of signs (both pluses). Let us denote $I = \sum_{k,l\geq 0} k^* l^* \varphi^2_{k,l}$. Then we can write (compare with the proof of Lemma 12)

$$\begin{aligned}
&\sum_{k,l\geq -1}\sum_{p,r\geq 1}(\psi^*_{k,l,p,r})^2 = \sum_{k,l\geq -1}\sum_{p,r\geq 1}\Big(\sum_{\substack{i\geq k+1\\ j\geq l+1}}\varphi_{i,j}\varphi_{i+p,j+r}\Big)^2\\
\leq\ & \sum_{k,l\geq -1}\sum_{\substack{i\geq k+1\\ j\geq l+1}}\varphi^2_{i,j}\sum_{p,r\geq 1}\sum_{\substack{m\geq k+1\\ n\geq l+1}}\varphi^2_{m+p,n+r} \leq \Big(\sum_{k,l\geq -1}\sum_{\substack{i\geq k+1\\ j\geq l+1}}\varphi^2_{i,j}\Big)\sum_{p,r\geq 1}\sum_{\substack{m\geq 0\\ n\geq 0}}\varphi^2_{m+p,n+r}\\
\leq\ & I\sum_{p,r\geq 1}\sum_{\substack{m\geq p\\ n\geq r}}\varphi^2_{m,n} \leq I^2.
\end{aligned}$$

Now we take both minuses, and it turns out that this case is much more complicated. In the first version of the paper a simple proof was incorrect, and we even need a stronger condition (comparing with the case with both pluses) on the coefficients $\varphi_{i,j}$. Recall that $\varphi_{i,j} = 0$ if $i < 0$ or $j < 0$, and, therefore,

$$\sum_{\substack{i\geq k+1\\ j\geq l+1}}\varphi_{i,j}\varphi_{i-p,j-r} = \sum_{\substack{i\geq \max(k+1,p)\\ j\geq \max(l+1,r)}}\varphi_{i,j}\varphi_{i-p,j-r}.$$

Then

$$\begin{aligned}
J\ &=\ \sum_{k,l\geq -1}\sum_{p,r\geq 1}(\psi^*_{k,l,-p,-r})^2 = \sum_{k,l\geq -1}\sum_{p,r\geq 1}\Big(\sum_{\substack{i\geq \max(k+1,p)\\ j\geq \max(l+1,r)}}\varphi_{i,j}\varphi_{i-p,j-r}\Big)^2\\
&=\ J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

where $J_i,\ i = 1,\dots,4$, are obtained by dividing the summation region $\{p, r \geq 1\}$ into four regions $\{1 \leq p \leq k+1, 1 \leq r \leq l+1\}$, $\{1 \leq p \leq k+1, r >$

$l+1\}$, $\{p > k+1, 1 \le r \le l+1\}$, $\{p > k+1, r > l+1\}$, respectively. Each term is estimated differently. Applying Hölder's inequality, we get

$$\begin{aligned} J_1 &= \sum_{k,l\ge -1} \sum_{p=0}^{k+1} \sum_{r=0}^{l+1} \Big(\sum_{\substack{i\ge k+1 \\ j\ge l+1}} \varphi_{i,j}\varphi_{i+p,j+r} \Big)^2 \qquad (23) \\ &\le \sum_{k,l\ge -1} \sum_{p=0}^{k+1} \sum_{r=0}^{l+1} \sum_{\substack{i\ge k+1 \\ j\ge l+1}} \varphi_{i,j}^2 \sum_{\substack{i\ge k+1 \\ j\ge l+1}} \varphi_{i-p,j-r}^2 \\ &\le \sum_{k,l\ge -1} \sum_{\substack{i\ge k+1 \\ j\ge l+1}} \varphi_{i,j}^2 c(k,l), \end{aligned}$$

where

$$c(k,l) = \sum_{p=0}^{k+1} \sum_{r=0}^{l+1} \sum_{\substack{i\ge k+1 \\ j\ge l+1}} \varphi_{i-p,j-r}^2.$$

If for a fixed l, we denote

$$c_1(k,l) = \sum_{i\ge k+1} \sum_{p=0}^{k+1} \varphi_{i-p,l}^2,$$

then it is easy to see that

$$\begin{aligned} c_1(k,l) &= \sum_{i\ge k+1} \varphi_{i,l}^2 + \sum_{i\ge k} \varphi_{i,l}^2 + \cdots + \sum_{i\ge 0} \varphi_{i,l}^2 \\ &\le \sum_{k\ge -1} \sum_{i\ge k+1} \varphi_{i,l}^2 \le \sum_{k\ge 0} \sum_{i\ge k+1} \varphi_{i,l}^2 + \sum_{i\ge 0} \varphi_{i,l}^2 \le \sum_{i\ge 0} i^* \varphi_{i,l}^2. \end{aligned}$$

Therefore, using the estimate for $c_1(k,l)$ with $l = j-r$ and then using the same estimate for the sum $\sum_{j\ge l+1} \sum_{r=0}^{l+1} \varphi_{i,j-r}^2$, we get

$$\begin{aligned} c(k,l) &= \sum_{j\ge l+1} \sum_{r=0}^{l+1} c_1(k, j-r) \le \sum_{j\ge l+1} \sum_{r=0}^{l+1} \sum_{i\ge 0} i^* \varphi_{i,j-r}^2 \\ &\le \sum_{i\ge 0} i^* \sum_{j\ge l+1} \sum_{r=0}^{l+1} \varphi_{i,j-r}^2 \le \sum_{i\ge 0} \sum_{j\ge 0} i^* j^* \varphi_{i,j}^2 = I. \end{aligned}$$

Having this estimate, from (23) we easily get

$$J_1 \le I \sum_{k,l\ge -1} \sum_{\substack{i\ge k+1\\ j\ge l+1}} \varphi_{i,j}^2 \le I \sum_{i\ge 0}\sum_{j\ge 0} i^* j^* \varphi_{i,j}^2 = I^2. \tag{24}$$

We estimate J_4 in a different way. Without loss of generality we may assume that, for all $k, l \ge 0$, $\varphi_{k,l} \ne 0$. Taking $\beta = (1+\varepsilon)/2 > 1/2$, where $\varepsilon > 0$ is from conditions of the lemma, and using (10), we can write

$$\begin{aligned} J_4 &= \sum_{k,l\ge -1} \sum_{\substack{p\ge k+1\\ r\ge l+1}} \Big(\sum_{\substack{i\ge p\\ j\ge r}} \varphi_{i,j}\varphi_{i-p,j-r} \Big)^2 \qquad (25)\\ &= \sum_{k,l\ge -1} \sum_{\substack{p\ge k+1\\ r\ge l+1}} \Big(\sum_{\substack{i\ge 0\\ j\ge 0}} \varphi_{i,j}\varphi_{i+p,j+r} \Big)^2 \\ &= \sum_{k,l\ge -1} \sum_{\substack{p\ge k+1\\ r\ge l+1}} \Big(\sum_{\substack{i\ge 0\\ j\ge 0}} (i^*j^*)^{2\beta} \varphi_{i,j}^2 \Big)\Big(\sum_{\substack{i\ge 0\\ j\ge 0}} \frac{\varphi_{i+p,j+r}^2}{(i^*j^*)^{2\beta}} \Big) \\ &\le \bar{C} I_\beta \sum_{k,l\ge -1} \sum_{\substack{p\ge k+1\\ r\ge l+1}} \varphi_{p,r}^2 \sum_{\substack{i\ge 0\\ j\ge 0}} \frac{1}{(i^*j^*)^{2\beta}} \le \bar{C} C_\beta^2 I_\beta I, \end{aligned}$$

where $I_\beta = \sum_{k,l\ge 0} (k^*l^*)^{2\beta}\varphi_{k,l}^2$ is finite due to the condition $\mathcal{L}_{1+\varepsilon,2}$ and $C_\beta = \sum_{j\ge 0} (j^*)^{-2\beta}$ is finite due to $2\beta > 1$.

Finally, the quantities J_2 and J_3 are estimated combining methods used in estimating J_1 and J_4. Let us take J_2. Applying Hölder's inequality, we get

$$\begin{aligned} J_2 &\le \sum_{k,l\ge -1} \sum_{p=0}^{k+1} \sum_{r=l+1}^{\infty} \Big(\sum_{\substack{i\ge k+1\\ j\ge 0}} \frac{\varphi_{i,j+r}^2}{(j^*)^{2\beta}} \Big)\Big(\sum_{\substack{i\ge k+1\\ j\ge 0}} (j^*)^{2\beta} \varphi_{i-p,j}^2 \Big) \qquad (26)\\ &= \sum_{k\ge -1} \sum_{p=0}^{k+1} \sum_{\substack{i\ge k+1\\ j\ge 0}} (j^*)^{2\beta}\varphi_{i-p,j}^2 \sum_{l\ge -1} \sum_{r=l+1}^{\infty} \sum_{\substack{i\ge k+1\\ j\ge 0}} \frac{\varphi_{i,j+r}^2}{(j^*)^{2\beta}}. \end{aligned}$$

Recalling that $2\beta > 1$ and using (10), we have

$$c_2(k) := \sum_{l\ge -1} \sum_{r=l+1}^{\infty} \sum_{\substack{i\ge k+1\\ j\ge 0}} \frac{\varphi_{i,j+r}^2}{(j^*)^{2\beta}} \le \bar{C} \sum_{l\ge -1} \sum_{r=l+1}^{\infty} \sum_{i\ge k+1} \varphi_{i,r}^2 \sum_{j\ge 0} (j^*)^{-2\beta}$$

$$\leq \bar{C}C_\beta \sum_{i\geq k+1}\sum_{j\geq 0} j^*\varphi_{i,j}^2.$$

Therefore, from (26) we get

$$\begin{aligned} J_2 &\leq \bar{C}C_\beta \sum_{k\geq -1}\sum_{p=0}^{k+1}\sum_{\substack{i\geq k+1\\ j\geq 0}} (j^*)^{2\beta}\varphi_{i-p,j}^2 \sum_{i\geq k+1}\sum_{j\geq 0} j^*\varphi_{i,j}^2 \qquad (27)\\ &\leq \bar{C}C_\beta \sum_{k\geq -1}\sum_{\substack{j\geq 0\\ \sum_{j\geq 0}}} i^*(j^*)^{2\beta}\varphi_{i,j}^2 \sum_{\substack{j\geq 0\\ \sum_{i\geq k+1}}} j^*\varphi_{i,j}^2 \leq \bar{C}C_\beta I_{1,\beta}\sum_{k\geq -1}\sum_{i\geq k+1}\sum_{j\geq 0} j^*\varphi_{i,j}^2\\ &\leq \bar{C}C_\beta I_{1,\beta} I, \end{aligned}$$

where $I_{1,\beta} = \sum_{j\geq 0} i^*(j^*)^{2\beta}\varphi_{i,j}^2$.

We have shown how to estimate the first quantity in (20) with two combinations of signs, both pluses or both minuses. The remaining two combinations of signs are dealt similarly, and we omit these calculations.

Thus, we have proved (20). Since the case for the corresponding coefficients with $p = r = 0$ is much more simple, we leave the proof of (21) for the reader.

Now let us denote $I_1 = \sum_{k,l\geq 0}(k^*l^*)^{1/2}\varphi_{k,l}^2$. We can write (again compare with the proof of Lemma 12)

$$\begin{aligned} \sum_{p,r\geq 1}\mu_{p,r}^2 &= \sum_{p,r\geq 1}\Big(\sum_{\substack{k\geq 0\\ l\geq 0}}\varphi_{k,l}\varphi_{k+p,l+r}\Big)^2\\ &= \sum_{p,r\geq 1}\Big(\sum_{k,l\geq 0}(k^*l^*)^{1/4}\varphi_{k,l}(k^*l^*)^{-1/4}\varphi_{k+p,l+r}\Big)^2\\ &\leq \big(\sum_{k,l\geq 0}(k^*l^*)^{1/2}\varphi_{k,l}^2\big)\sum_{p,r\geq 1}\sum_{k,l\geq 0}(k^*l^*)^{-1/2}\varphi_{k+p,l+r}^2\\ &= I_1\sum_{k,l\geq 0}(k^*l^*)^{-1/2}\sum_{\substack{i\geq k+1\\ j\geq l+1}}\varphi_{i,j}^2 \leq I_1^2. \end{aligned}$$

Since it is easy to see that $\sum_{p,r\geq 1}\mu_{-p,-r}^2 = \sum_{p,r\geq 1}\mu_{p,r}^2$, we now show how to estimate the quantity with different signs. We have

$$\begin{aligned}\sum_{p,r\geq 1}\mu_{p,-r}^2 &= \sum_{p,r\geq 1}\Big(\sum_{\substack{k\geq 0\\ l\geq r}}\varphi_{k,l}\varphi_{k+p,l-r}\Big)^2 = \sum_{p,r\geq 1}\Big(\sum_{k,l\geq 0}\varphi_{k,l+r}\varphi_{k+p,l}\Big)^2\\ &= \sum_{p,r\geq 1}\Big(\sum_{k,l\geq 0}k^{*1/4}l^{*-1/4}\varphi_{k,l+r}k^{*-1/4}l^{*1/4}\varphi_{k+p,l}\Big)^2\\ &\leq \sum_{p\geq 1}\Big(\sum_{k,l\geq 0}k^{*1/2}l^{*-1/2}\varphi_{k,l+r}^2\Big)\sum_{r\geq 1}\Big(\sum_{k,l\geq 0}k^{*-1/2}l^{*1/2}\varphi_{k+p,l}^2\Big).\end{aligned}$$

It is not difficult to get the estimate

$$\sum_{p\geq 1}\Big(\sum_{k,l\geq 0}k^{*1/2}l^{*-1/2}\varphi_{k,l+r}^2\Big)\leq I_1$$

and similarly

$$\sum_{r\geq 1}\Big(\sum_{k,l\geq 0}k^{*-1/2}l^{*1/2}\varphi_{k+p,l}^2\Big)\leq I_1.$$

Therefore, we get

$$\sum_{p,r\geq 1}\mu_{p,-r}^2<\infty$$

if I_1 is finite. Similarly, we can deal with another combination of signs, and thus relation (22) is proved. The lemma is proved.

4 Proofs of main results

Proof of Theorem 2. As was mentioned, relation (4) (more precisely, relation (12)) in a slightly different form was proved in [10], and the proof of (5) is given in Lemma 12.

Proof of Proposition 4. Using (4), we can write

$$X_{t,s}=\Phi(\mathbf{L})\varepsilon_{t,s}=(\mu_1+A_2(\mathbf{L})-A_1(\mathbf{L}))\varepsilon_{t,s}.$$

Summing these equalities over D_n, we get (6) with

$$R_n=\sum_{t,s\in D_n}(A_2(\mathbf{L})-A_1(\mathbf{L}))\varepsilon_{t,s}.$$

It remains to show that this expression can be written as it was stated in the proposition. We start with the term

$$R_{n,1}=\sum_{t,s\in D_n}A_2(\mathbf{L})\varepsilon_{t,s}=\sum_{t,s\in D_n}\Delta_2(\mathbf{L})\xi_{t,s}.$$

Let us denote $\Delta_2^{(j)}(\mathbf{L}) = (1 - L_1^j)(1 - L_2^j),\ j \geq 2,\ \Delta_2^{(1)}(\mathbf{L}) = \Delta_2(\mathbf{L})$. It is easy to verify that

$$\Delta_2(\mathbf{L})\xi_{t,s} + \Delta_2(\mathbf{L})\xi_{t-1,s} + \Delta_2(\mathbf{L})\xi_{t,s-1} + \Delta_2(\mathbf{L})\xi_{t-1,s-1} = \Delta_2^{(2)}(\mathbf{L})\xi_{t,s},$$

$$\begin{aligned} \Delta_2^{(2)}(\mathbf{L})\xi_{t,s} &+ \Delta_2(\mathbf{L})\xi_{t-2,s} + \Delta_2(\mathbf{L})\xi_{t-2,s-1} + \Delta_2(\mathbf{L})\xi_{t-2,s-2} \\ &+ \Delta_2(\mathbf{L})\xi_{t-1,s-2} + \Delta_2(\mathbf{L})\xi_{t,s-2} = \Delta_2^{(3)}(\mathbf{L})\xi_{t,s}, \end{aligned}$$

and so on. Therefore, starting this process from the element $\xi_{n,n}$, we get

$$R_{n,1} = \sum_{t,s \in D_n} \Delta_2(\mathbf{L})\xi_{t,s} = \Delta_2^{(n)}(\mathbf{L})\xi_{n,n} = \xi_{n,n} - \xi_{n,0} - \xi_{0,n} + \xi_{0,0}. \tag{28}$$

Now, using the notation introduced in the proposition, we have

$$\begin{aligned} R_{n,2} &= \sum_{t,s \in D_n} B(L_1)(1 - L_1)\varepsilon_{t,s} = \sum_{t=1}^{n}\sum_{s=1}^{n}(\bar{\varepsilon}_{t,s} - \bar{\varepsilon}_{t-1,s}) \\ &= \sum_{t=1}^{n}(\eta_{t,n} - \eta_{t-1,n}) = \eta_{n,n} - \eta_{0,n}. \end{aligned} \tag{29}$$

Similarly, we get

$$R_{n,3} = \sum_{t,s \in D_n} D(L_2)(1 - L_2)\varepsilon_{t,s} = \zeta_{n,n} - \zeta_{n,0}. \tag{30}$$

Since $R_n = \sum_{i=1}^{3} R_{n,i}$, (28)–(30) prove the proposition.

Proof of Theorem 5. Z_n is a sum of i.i.d. random variables with finite second moments; therefore, under an appropriate normalizing, both SLLN and CLT for Z_n hold, and to prove the theorem, we need to show that

$$n^{-2}R_n \xrightarrow{a.s.} 0 \qquad (\text{for } SLLN) \tag{31}$$

and

$$n^{-1}R_n \xrightarrow{P} 0 \qquad (\text{for } CLT). \tag{32}$$

We start with the proof of (31). Since $E\varepsilon_{00}^2 = 1$ and we have $\mathcal{L}_2$, from Lemma 12 it follows that $\xi_{t,s} = \Phi^*(\mathbf{L})\varepsilon_{t,s}$ is a stationary random field and $E\xi_{t,s}^2$ is finite and constant for all t, s. Thus, trivially,

$$n^{-2}\xi_{0,0} \xrightarrow{a.s.} 0, \tag{33}$$

and since

$$\sum_{n=1}^{\infty} P(|n^{-2}\xi_{n,n}| > \epsilon) \le E\xi_{1,1}^2 \sum_{n=1}^{\infty} \epsilon^{-2} n^{-4} < \infty,$$

we get

$$n^{-2}\xi_{n,n} \xrightarrow{a.s.} 0. \tag{34}$$

Similarly, we prove

$$n^{-2}\xi_{0,n} \xrightarrow{a.s.} 0, \quad n^{-2}\xi_{n,0} \xrightarrow{a.s.} 0. \tag{35}$$

To prove that

$$n^{-2}\eta_{n,n} \xrightarrow{a.s.} 0, \tag{36}$$

we use the fact that, due to the condition $\sum_{j\ge 0} |b_j|^2 < \infty$ proved in Lemma 12, $\bar{\varepsilon}_{t,s}$ is a stationary field with mean zero and finite second moment; moreover, $\bar{\varepsilon}_{t,s}$ and $\bar{\varepsilon}_{t,v}$ are independent for $s \ne v$. Then

$$P(n^{-2}\Big|\sum_{s=1}^{n} \bar{\varepsilon}_{n,s}\Big| > \epsilon) \le \epsilon^{-2} n^{-4} \sum_{s=1}^{n} E\bar{\varepsilon}_{n,s}^2 \le C\epsilon^{-2} n^{-3},$$

and, therefore, (36) follows. Similarly, we prove

$$n^{-2}\eta_{n,0} \xrightarrow{a.s.} 0, \quad n^{-2}\zeta_{n,n} \xrightarrow{a.s.} 0, \quad n^{-2}\zeta_{n,0} \xrightarrow{a.s.} 0. \tag{37}$$

From (33)–(37) we get (31). Using the same Tchebyshev's inequalities as in the proof of (31), we get (32). The theorem is proved.

Proof of Theorem 6. Again, as in the proof of Theorem 5, we must prove (31) under the weaker moment assumption. Since $\varepsilon_{t,s}$ are i.i.d. mean-zero random variables, applying the well-known moment inequality with $1 < p = 1 + \beta < 2$ for $\xi_{t,s} = \sum_{k,l\ge 0} \varphi_{k,l}^* \varepsilon_{t-k,s-l}$, we get

$$E|\xi_{t,s}|^p \le C \sum_{k,l\ge 0} |\varphi_{k,l}^*|^p E|\varepsilon_{t-k,s-l}|^p \le C \sum_{k,l\ge 0} |\varphi_{k,l}^*|^p < \infty.$$

Hence,

$$\sum_{n=1}^{\infty} P(|n^{-2}\xi_{n,n}| > \epsilon) \le E|\xi_{1,1}|^p \sum_{n=1}^{\infty} \epsilon^{-p} n^{-2p} < \infty,$$

and we have (34). Similarly, we get (35) and (33). Applying the same moment inequality, we have

$$P(n^{-2}|\eta_{n,n}| > \epsilon) \le \epsilon^{-p} n^{-2p} E|\eta_{n,n}|^p \le C\epsilon^{-p} n^{-(1+2\beta)} E|\bar{\varepsilon}_{n,1}|^p. \tag{38}$$

Using the condition $\mathcal{L}_{1+\beta}$ and once more the moment inequality, we have

$$E|\bar{\varepsilon}_{n,1}|^p \le CE|\varepsilon_{1,1}|^p \sum_{j\ge 0} |b_j|^p < \infty.$$

Therefore,

$$\sum_{n=1}^{\infty} P(|n^{-2}\eta_{n,n}| > \epsilon) < \infty,$$

and we have (36). Similarly, we get (37), and all these relations give us (31).

We believe that the theorem also remains valid for $\beta = 0$; in the estimate of $\xi_{n,n}$, we can take $\beta = 0$ with still remaining the convergent series, while if we take $\beta = 0$ in (38), we get the divergent harmonic series $\sum_{n=1} n^{-1}$.

Proof of Theorem 7. Since under the assumptions of the theorem we have a limit theorem for Z_n, as in the proof of Theorem 5, we need to show that

$$n^{-2/\alpha} R_n \xrightarrow{P} 0.$$

The assumption on $\varepsilon_{1,1}$ means that

$$P(|\varepsilon_{1,1}| > x) \sim Cx^{-\alpha} \quad as \quad x \to \infty.$$

From Lemma 12 we have that $\sum_{k,l\ge 0} |\varphi^*_{k,l}|^\alpha < \infty$; therefore, applying results on the tail behavior of weighted series of i.i.d. random variables (see Lemmas A3 and A4 in [12]), we get

$$P(|\xi_{n,n}| > x) \sim C \sum_{k,l\ge 0} |\varphi^*_{k,l}|^\alpha x^{-\alpha} \quad as \quad x \to \infty.$$

Hence,

$$n^{-2/\alpha} \xi_{n,n} \xrightarrow{P} 0.$$

In the same way, taking into account that, by Lemma 12, $\sum_{j\ge 0} |b_j|^\alpha < \infty$, we get the relation

$$P(|\bar{\varepsilon}_{n,s}| > x) \sim C \sum_{j\ge 0} |b_j|^\alpha x^{-\alpha} \quad as \quad x \to \infty.$$

Therefore, $n^{-1/\alpha}\eta_{n,n}$ converges to some stable law, and thus

$$n^{-2/\alpha} \eta_{n,n} \xrightarrow{P} 0.$$

The same arguments apply to the term with $\zeta_{n,n}$ and to other terms in R_n. The theorem is proved.

Proof of Theorem 8. Now we do not use BND directly but only the fact that $X_{t,s}$ is approximated by $\mu_1 \varepsilon_{t,s}$. Let us denote

$$\omega_n = S_n - \mu_1 Z_n.$$

It is easy to see that to prove CLT for S_n it suffices to show that

$$n^{-2} E\omega_n^2 \to 0. \tag{39}$$

We want to apply Lemma 13, and, for this reason, we again use the following agreement: $\varphi_{k,l} = 0$ if $k < 0$ or $l < 0$. Then it is not difficult to see that

$$\omega_n = S_n - \mu_1 Z_n = \sum_{(i,j)\in \mathbf{Z}^2} \sum_{t=1}^{n} \sum_{s=1}^{n} b_{t-i,s-j} \varepsilon_{i,j},$$

where $b_{0,0} = \varphi_{0,0} - \mu_1$ and $b_{k,l} = \varphi_{k,l}$ if $(k,l) \neq (0,0)$. Then

$$E\omega_n^2 = E\varepsilon_{0,0}^2 \sum_{(i,j)\in \mathbf{Z}^2} \Big(\sum_{t=1}^{n} \sum_{s=1}^{n} b_{t-i,s-j} \Big)^2;$$

therefore, recalling that $|\mathbf{n}| = n^2$ in our case and applying Lemma 13, we get (39).

Since SLLN for Z_n easily follows, to prove SLLN for S_n, one needs to show the convergence of the series

$$\sum_{n=1}^{\infty} P(n^{-2}|\omega_n| \geq \epsilon),$$

which, after applying elementary moment inequality with $1 < p \leq 2$, will converge if

$$\sum_{n=1}^{\infty} n^{-2p} E|\omega_n|^p < \infty. \tag{40}$$

Again applying Lemma 13, we get

$$n^{-2} E|\omega_n|^p \to 0,$$

and since $2p - 2 > 1$ if $p > 3/2$, we finally get (40). The theorem is proved.

Proof of Proposition 9. Since the proof of this technical proposition is lengthy, but not complicated, we present only the main steps, omitting the details. Taking the squares of both sides of equality (3), it is not difficult to get

$$X_{t,s}^2 = \Psi_{0,0}(\mathbf{L})\varepsilon_{t,s}^2 + \textstyle\sum^*_{p,r\geq 1}\Psi_{\pm p,\pm r}(\mathbf{L})\varepsilon_{t,s}\varepsilon_{t\mp p,s\mp r},$$

where $\sum^*$, as in (8), means that there are four sums with all possible combinations of signs. Here and in what follows, we use the convention that $\varphi_{k,l} = 0$ if $k < 0$ or $l < 0$. Now for each term $\Psi_{\pm p,\pm r}(\mathbf{L})$, we apply BND (4), for example,

$$\Psi_{0,0}(\mathbf{L}) = \mu_2 + A_{0,0}(\mathbf{L}) - B_{0,0}(L_1) - C_{0,0}(L_2),$$

and we easily get (8). Summing these relations over D_n, we get (9), and it remains to describe the structure of the remainder term $R_{n,2}$, which formally can be written as

$$\textstyle\sum_{(t,s)\in D_n}\sum^{**}_{p,r\geq 1}(\mu_{\pm p,\pm r} + A_{\pm p,\pm r}(\mathbf{L}) - B_{\pm p,\pm r}(L_1) - C_{\pm p,\pm r}(L_2))\varepsilon_{t,s}\varepsilon_{t\mp p,s\mp r},$$

where $\sum^{**}$ means that with four sums there is additional term with $p = r = 0$. Due to the presence of operators $\Delta_2(\mathbf{L})$ and $\Delta(L_i)$, this expression can be simplified similarly as it was done when deriving relation (7). We shall show that

$$R_{n,2} = J_0 + \sum_{i=1}^{4}\sum_{j=1}^{4} J_i^{(j)}, \tag{41}$$

where the index j corresponds to different combinations of signs $\pm$, and the index i corresponds to the part of the remainder obtained from different terms. The terms $J_1^{(j)}$ are obtained by summing the quantities with $\mu_{\pm p,\pm r}$, for example,

$$J_1^{(1)} = \sum_{(t,s)\in D_n}\sum_{p,r\geq 1}\mu_{p,r}\varepsilon_{t,s}\varepsilon_{t-p,s-r}, \quad J_1^{(2)} = \sum_{(t,s)\in D_n}\sum_{p,r\geq 1}\mu_{p,-r}\varepsilon_{t,s}\varepsilon_{t-p,s+r},$$

and so on. The terms $J_2^{(j)}$ are obtained by summing the quantities with $A_{\pm p,\pm r}(\mathbf{L})$. Let us denote

$$\xi_{t,s,\pm p,\pm r} = \Psi^*_{\pm p,\pm r}(\mathbf{L})\varepsilon_{t,s}\varepsilon_{t\mp p,s\mp r} = \sum_{k,l\geq 0}\psi^*_{k,l,\pm p,\pm r}\varepsilon_{t-k,s-l}\varepsilon_{t-k\mp p,s-l\mp r}$$

and by $\xi_{t,s}^{(j)}$, $j = 1,\ldots,4$, the sums $\sum_{p,r\geq 1}\xi_{t,s,\pm p,\pm r}$ with appropriate combinations of signs. We also denote $\xi_{t,s}^{(5)} = \xi_{t,s,0,0}$. Using the operators $\Delta_2^{(j)}(\mathbf{L})$ as in the proof of Proposition 4, we get

$$J_2^{(j)} = \sum_{(t,s)\in D_n}\Delta_2(\mathbf{L})\xi_{t,s}^{(j)} = \Delta_2^{(n)}(\mathbf{L})\xi_{n,n}^{(j)} = \xi_{n,n}^{(j)} - \xi_{n,0}^{(j)} - \xi_{0,n}^{(j)} + \xi_{0,0}^{(j)}.$$

The terms $J_3^{(j)}$ and $J_4^{(j)}$ are obtained by summing the quantities with $B_{\pm p,\pm r}(1-L_1)$ and $C_{\pm p,\pm r}(1-L_2)$, respectively. To write down these terms, we need more notation. We set

$$\bar{\varepsilon}_{t,s,\pm p,\pm r} = \sum_{j\geq 0} b_{j,\pm p,\pm r}\varepsilon_{t-j,s}\varepsilon_{t-j\mp p,s\mp r},$$

$$\hat{\varepsilon}_{t,s,\pm p,\pm r} = \sum_{j\geq 0} d_{j,\pm p,\pm r}\varepsilon_{t,s-j}\varepsilon_{t\mp p,s-j\mp r},$$

and in the obvious way we define $\bar{\varepsilon}_{t,s}^{(j)}$, $\hat{\varepsilon}_{t,s}^{(j)}$, $j = 1,2,3,4$, $\bar{\varepsilon}_{t,s}^{(5)}$, $\hat{\varepsilon}_{t,s}^{(5)}$, for example,

$$\bar{\varepsilon}_{t,s}^{(1)} = \sum_{p,r\geq 1} \bar{\varepsilon}_{t,s,p,r}, \quad \bar{\varepsilon}_{t,s}^{(4)} = \sum_{p,r\geq 1} \bar{\varepsilon}_{t,s,-p,-r}.$$

Again as in the proof of Proposition 4, we get

$$J_3^{(j)} = \sum_{(t,s)\in D_n} (1-L_1)\bar{\varepsilon}_{t,s}^{(j)} = \sum_{t=1}^{n}(\eta_{t,n}^{(j)} - \eta_{t-1,n}^{(j)}) = \eta_{n,n}^{(j)} - \eta_{0,n}^{(j)},$$

$$J_4^{(j)} = \sum_{(t,s)\in D_n} (1-L_2)\hat{\varepsilon}_{t,s}^{(j)} = \sum_{s=1}^{n}(\zeta_{n,s}^{(j)} - \eta_{n,s-1}^{(j)}) = \zeta_{n,n}^{(j)} - \zeta_{n,0}^{(j)},$$

where

$$\eta_{t,n}^{(j)} = \sum_{s=1}^{n} \bar{\varepsilon}_{t,s}^{(j)}, \quad \zeta_{n,s}^{(j)} = \sum_{t=1}^{n} \hat{\varepsilon}_{t,s}^{(j)}.$$

Finally, the term J_0 corresponds to the case $p = r = 0$:

$$\begin{aligned} J_0 &= \sum_{(t,s)\in D_n} (A_{0,0}(\mathbf{L}) - B_{0,0}(1-L_1) - C_{0,0}(1-L_2))\varepsilon_{t,s}^2 \\ &= \Delta_2^{(n)}(\mathbf{L})\xi_{n,n}^{(5)} - (\eta_{n,n}^{(5)} - \eta_{0,n}^{(5)}) - (\zeta_{n,n}^{(5)} - \zeta_{n,0}^{(5)}). \end{aligned}$$

We have written all terms in (41), and the proposition is proved.

Proof of Theorem 10. From (9) and (41) we see that to prove the theorem one needs to show that all 17 terms in (41) divided by n^2 tend to zero a.s. Since all terms $J_i^{(j)}$, with different j for a fixed i are very similar, we shall show only relations for $j = 1$:

$$n^{-2}J_i^{(1)} \xrightarrow{a.s.} 0. \tag{42}$$

We start with the proof of the relation

$$n^{-2}J_1^{(1)} \xrightarrow{a.s.} 0. \qquad (43)$$

Let us denote

$$\varepsilon^f_{t-1,s-1} = \sum_{p,r\geq 1} \mu_{p,r}\varepsilon_{t-p,s-r}.$$

Due to Lemma 15,

$$E(\varepsilon^f_{t-1,s-1})^2 = \sum_{p,r\geq 1} \mu^2_{p,r} < \infty,$$

and ε^f is a mean-zero stationary random field; moreover, $\varepsilon_{t,s}$ and $\varepsilon^f_{t-1,s-1}$ are independent. Therefore, we have

$$n^{-4}E(J_1^{(1)})^2 = n^{-4}E\Big(\sum_{s,t=1}^{n} \varepsilon_{t,s}\varepsilon^f_{t-1,s-1}\Big)^2 = n^{-4}\sum_{s,t=1}^{n} E\varepsilon^2_{t,s}E(\varepsilon^f_{t-1,s-1})^2 \leq Cn^{-2},$$

since $E\varepsilon_{t,s}\varepsilon^f_{t-1,s-1}\varepsilon_{u,v}\varepsilon^f_{u-1,v-1} = 0$ if $t \neq u$ or $s \neq v$. Hence, for any $\epsilon > 0$,

$$\sum_{n\geq 1} P(n^{-2}|J_1^{(1)}| \geq \epsilon) < \infty,$$

and (43) follows. Let us note that, for other values of j, we define in appropriate way the random fields $\varepsilon^f_{t\pm 1,s\pm 1} = \sum_{p,r\geq 1} \mu_{\pm p,\pm r}\varepsilon_{t\mp p,s\mp r}$ and use the condition

$$\sum_{p,r\geq 1} \mu^2_{\pm p,\pm r} < \infty.$$

To prove the relation

$$n^{-2}J_2^{(1)} \xrightarrow{a.s.} 0, \qquad (44)$$

consider

$$\xi^{(1)}_{n,n} = \sum_{p,r\geq 1}\sum_{k,l\geq 0} \psi^*_{k,l,p,r}\varepsilon_{n-k,n-l}\varepsilon_{n-k-p,n-l-r}.$$

It is easy to see that, due to Lemma 15,

$$E(\xi^{(1)}_{n,n})^2 = \sum_{p,r\geq 1}\sum_{k,l\geq 0} (\psi^*_{k,l,p,r})^2 E\varepsilon^2_{n-k,n-l}E\varepsilon^2_{n-k-p,n-l-r}$$

is finite and does not depend on n. Thus, we get

$$n^{-2}\xi^{(1)}_{n,n} \xrightarrow{a.s.} 0,$$

and since the same relation for $\xi_{n,0}^{(1)}$, $\xi_{0,n}^{(1)}$, $\xi_{0,0}^{(1)}$ can be proved with minor changes, we get (44). To get this relation for $j = 2, 3, 4$, we must consider other combinations of signs in the expression

$$\sum_{p,r\geq 1}\sum_{k,l\geq 0}\psi^*_{k,l,\pm p,\pm r}\varepsilon_{n-k,n-l}\varepsilon_{n-k\mp p,n-l\mp r}$$

and use the appropriate conditions from Lemma 15. Now consider

$$n^{-2}\eta_{n,n}^{(1)} = n^{-2}\sum_{s=1}^{n}\bar{\varepsilon}_{n,s}^{(1)}.$$

It is easy to note that $\bar{\varepsilon}_{n,s_1}^{(1)}$ and $\bar{\varepsilon}_{n,s_2}^{(1)}$ are independent for $s_1 \neq s_2$ and

$$E(\bar{\varepsilon}_{n,s}^{(1)})^2 = \sum_{p,r\geq 1}\sum_{j\geq 0}b_{j,p,r}^2$$

is finite due to Lemma 15; therefore,

$$n^{-4}E(\eta_{n,n}^{(1)})^2 = n^{-4}\sum_{s=1}^{n}E(\bar{\varepsilon}_{n,s}^{(1)})^2 \leq Cn^{-3}.$$

The same relation holds for $\eta_{0,n}^{(1)}$, and we easily get

$$n^{-2}J_3^{(1)} \xrightarrow{a.s.} 0 \tag{45}$$

In the same way we also prove

$$n^{-2}J_4^{(1)} \xrightarrow{a.s.} 0 \tag{46}$$

for the other values $j = 2, 3, 4$. A final remark concerns the proof of

$$n^{-2}J_0 \xrightarrow{a.s.} 0, \tag{47}$$

and this is the case $j = 5$ for $i = 2, 3, 4$, which is simpler since we have no sum over all $p, r \geq 1$ and set $p = r = 0$. Relations (43)–(47) give us (42), and the theorem is proved.

Proof of Theorem 11. Essentially, we repeat the proof of Theorem 8; therefore, we make only several remarks. We can take the one-to-one mapping $g : \{(t,s) \in Z^2,\ t,s \geq 1\} \to \mathbf{N}$ such that, denoting $\tilde{\varepsilon}_n = \varepsilon_{t,s}$ if $g(t,s) = n$ and $\tilde{Z}_n = \sum_{k=1}^{n}\tilde{\varepsilon}_k$, we get that $Z_n = \tilde{Z}_{n^2}$. However, $n^{-1/2}\tilde{Z}_n$ is a normalized sum of i.i.d. random elements with mean zero and finite second moment in a Banach space of type 2; therefore, CLT for this sequence holds (see, for example, [14]) with limit $N(0, A)$, where A is the covariance operator for $\tilde{\varepsilon}_1 = \varepsilon_{1,1}$. Since $n^{-1}Z_n$ is a subsequence of this weakly convergent sequence, we get CLT for $\mu_1 n^{-1} Z_n$. Then, instead of (39), we prove

$$n^{-2}E||\omega_n||^2 \to 0$$

using the moment inequality in a Banach space of type 2 (see, again, [14]) and Lemma 13 but now with the coefficients $b_{k,l} \in L(B)$. The theorem is proved.

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