Remarks on the SLLN for linear random fields

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Abstract

We consider random linear fields on $\mathbb{Z}^d$ generated by ergodic or mixing (in particular case, independent identically distributed (i.i.d.)) random variables. Our main results generalize classical Strong Law of Large Numbers (SLLN) for multi-indexed sums of i.i.d. random variables. These results are easily obtained using ergodic theory. Also we compare the results for SLLN obtained using ergodic theory and with the help of the Beveridge–Nelson decomposition.

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1 Introduction

Consider a linear random field

$$X_{\mathbf{t}} = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}} \varepsilon_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} = (t_1, \dots, t_d) \in \mathbb{Z}^d, \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d, \qquad (1)$$

where the coefficients $\{\varphi_{\mathbf{k}},\ \mathbf{k} \geqslant 0,\ \mathbf{k} \in \mathbb{Z}^d\}$ and random variables $\{\varepsilon_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ are such that the random field $\{X_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ is well defined and is stationary. Here and in what follows bold letters stand for vectors (multi-dimensional or infinite-dimensional), linear operations and inequalities are component-wise, for example, $\mathbf{1} = (1, \dots, 1)$, for $\mathbf{x}, \mathbf{y} \in R^d$ $\mathbf{x} \leqslant \mathbf{y}$ means that $x_i \leqslant y_i,\ i = 1, \dots, d$.

In the paper Phillips and Solo (1992) it was demonstrated that the so-called Beveridge–Nelson decomposition (BND) presents rather simple method for proving limit theorems (Central Limit Theorem (CLT), Strong Law of Large Numbers (SLLN), Law of Iterated Logarithm (LIL), and Invariance Principle (IP)) for sums of values of linear processes. In the recent paper Paulauskas (2009) it was demonstrated that the same BND for linear random fields is useful when proving limit theorems for sums $\sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}}$, where $D_{\mathbf{n}}$ is some subset of $\mathbb{Z}^d$. Namely, this representation (exact formulation in the case $d = 2$ see below) allows to write

$$\sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}} = M_1 \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}} + R_{\mathbf{n}},$$

where $M_1 = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}}$, and $R_{\mathbf{n}}$ in the case $d = 2$ and for simple sets $D_{\mathbf{n}}$ (rectangles or squares) has not complicated form. Thus, assuming that a linear field is generated by i.i.d. random variables $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$, we reduce the investigation of the limit behavior of sum of dependent stationary random variables to the same problem for sums of i.i.d. random variables and the problem to show that $R_{\mathbf{n}}$ (in some sense) converges to zero. In Paulauskas (2009) the BND for linear random fields was applied to prove CLT and SLLN, earlier in Marinucci and Poghosyan (2001) BND was used to prove IP. In Paulauskas (2009) it was stressed that in the case $d = 2$ and sets $D_n = [1, n]^2 \cap \mathbb{Z}^2$ this approach leads to very simple proofs, but at the same time moment conditions for innovations $\varepsilon_{\mathbf{t}}$ and conditions for coefficients $\varphi_{\mathbf{k}}$ are not optimal (this is the price which we pay for simplicity of proofs). In this short note we consider SLLN on rectangles

$$D_{\mathbf{n}} = \{\mathbf{t} \in \mathbb{Z}^d : 1 \leqslant t_i \leqslant n_i,\ i = 1, 2, \dots, d\}. \qquad (2)$$

and our goal is to obtain generalization of the classical SLLN for multi-indexed sums of i.i.d. random variables (see Smythe (1973)). The main message of our note is that the application of the ergodic theory to prove SLLN for linear fields gives much more general and stronger results comparing with ones obtained by using BND and even the proofs, based on application of ergodic theorems, are very simple. Since the application of BND for IP also faces some difficulties (see Paulauskas (2009) and Marinucci and Poghosyan (2001)), it seems that the most successful application of BND is for the CLT in Paulauskas (2009).

Before formulation of our results we introduce some notions from the ergodic theory. Let $Y = \{Y_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ be a strictly stationary random field. Let $\mathrm{H} = R^{\mathbb{Z}^d}$ denote a space of all real-valued functions on $\mathbb{Z}^d$ with a σ-algebra $\mathcal{H}$, generated by cylindrical sets. Denote by $\{U_{\mathbf{h}},\ \mathbf{h} \in \mathbb{Z}^d\}$ the group of translations:

$$U_{\mathbf{h}} x_{\mathbf{t}} = x_{\mathbf{t}-\mathbf{h}}, \quad x \in \mathrm{H},\ \mathbf{t}, \mathbf{h} \in \mathbb{Z}^d.$$

Let $\mathbf{P}$ stand for a distribution of a random field Y in H. Strict stationarity of Y means that $\mathbf{P}$ is invariant with respect to translations:

$$\mathbf{P} U_{\mathbf{h}}^{-1} = \mathbf{P}.$$

Let $\mathcal{T}$ denote σ-algebra of invariant sets:

$$\mathcal{T} = \{A \in \mathcal{H} :\ U_{\mathbf{h}}(A) = A,\ \forall\ \mathbf{h} \in \mathbb{Z}^d\}.$$

A random field is *ergodic* if σ-algebra $\mathcal{T}$ is trivial:

$$\forall\ A \in \mathcal{T} \quad \mathbf{P}(A) = 0 \text{ or } 1. \tag{3}$$

From the ergodic theory it follows that relation (3) is equivalent to the following relation:

$$\forall\ A, B \in \mathcal{H} \quad n^{-d} \sum_{\mathbf{0} \le \mathbf{h} \le \bar{\mathbf{n}} - \mathbf{1}} \mathbf{P}(A \cap U_{\mathbf{h}}^{-1}(B)) \to \mathbf{P}(A)\mathbf{P}(B), \tag{4}$$

as $n \to \infty$, here $\bar{\mathbf{n}} = (n, \ldots, n)$. For more information on ergodic theory see Georgii (1988) or Walters (1982).

We say that a random field Y is *mixing* if

$$\forall\ A, B \in \mathcal{H} \quad \mathbf{P}(A \cap U_{\mathbf{h}}^{-1}(B)) \to \mathbf{P}(A)\mathbf{P}(B), \tag{5}$$

as $||\mathbf{h}|| \to \infty$, here $||\cdot||$ stands for any norm in R^d. It is clear that (5) implies (4), thus, mixing implies ergodicity of a random field. Such definitions for measure-preserving transformations (but in case $d = 1$) can be found in Walters (1982).

While using BND approach we consider linear fields of the form (1), the ergodic theory approach allows us to consider more general linear random fields with summation extended not over positive quadrant bur over all $\mathbb{Z}^d$. Namely, let

$$Y_{\mathbf{t}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} \varphi_{\mathbf{k}} \varepsilon_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} \in \mathbb{Z}^d. \tag{6}$$

We consider sums

$$S_{\mathbf{n}} = \sum_{\mathbf{t} \in D_{\mathbf{n}}} Y_{\mathbf{t}}, \tag{7}$$

and let us denote $|\mathbf{n}| := \prod_{i=1}^d n_i$. We say that SLLN holds for $S_{\mathbf{n}}$, if

$$|\mathbf{n}|^{-1} S_{\mathbf{n}} \overset{a.s.}{\to} 0, \tag{8}$$

when $\mathbf{n}$ tends to infinity. There are several interpretations of the growth of $\mathbf{n}$. In our paper we shall use two possibilities. We shall write $\mathbf{n} \to \infty$ if

$$n_i \to \infty, \ i = 1, \dots, d. \tag{9}$$

The second possibility of growth of $\mathbf{n}$ is to assume that

$$|\mathbf{n}| \to \infty. \tag{10}$$

Evidently, (10) follows from (9), but not converse.

Let us denote by $\mathcal{L}_{q,p}$ condition

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \Big(\prod_{i=1}^d (|k_i| + 1) \Big)^q |\varphi_{\mathbf{k}}|^p < \infty$$

and $\mathcal{L}_p := \mathcal{L}_{p,p}$. If a random variable X satisfies

$$E|X| (\ln(1 + |X|))^{d-1} < \infty, \tag{11}$$

we shall write $X \in L \log L^{d-1}$.

Theorem 1. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ be a strictly stationary ergodic random field with $E\varepsilon_{\mathbf{0}} = 0$ and $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$. Suppose that condition $\mathcal{L}_{0,1}$ holds and $S_{\mathbf{n}}$ is defined in (7). Then, if $\mathbf{n} \to \infty$, the relation (8) holds.*

In particular, the result holds for the case where $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ are i.i.d. random variables satisfying the same moment conditions . Therefore in the case where $Y_{\mathbf{t}} = \varepsilon_{\mathbf{t}}$ (this will be if $\varphi_{\mathbf{0}} = 1$, $\varphi_{\mathbf{k}} = 0$ for all $\mathbf{k} \neq \mathbf{0}$) we get the generalization of the result of Smythe (1973) where it is shown that if $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ are i.i.d. random variables, then SLLN for $S_{\mathbf{n}} = \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}}$ holds if an only if (11) is satisfied. If we consider the class of all strictly stationary ergodic random fields, which includes a random field $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ with i.i.d. random variables $\varepsilon_{\mathbf{t}}$, therefore moment condition $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$ in our theorem is necessary, too. But as results of Rieders (1993) show, for particular types of dependent stationary sequences situation with necessary conditions can be different, the same can be said about random fields.

If we require stronger condition on initial random field $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, we get a stronger result.

Theorem 2. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ be a strictly stationary mixing random field with $E\varepsilon_{\mathbf{0}} = 0$ and $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$. Suppose that condition $\mathcal{L}_{0,1}$ holds. Then the relation* (8) *holds, if $|\mathbf{n}| \to \infty$.*

Both formulated theorems rely on the classical ergodic theorem of Zygmund-Fava.

Theorem A. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ be a strictly stationary ergodic random field with $E\varepsilon_{\mathbf{0}} = 0$ and $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$. Then*

$$V_{\mathbf{n}} = |\mathbf{n}|^{-1} \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}} \xrightarrow{a.s.} 0,$$

as $\mathbf{n} \to \infty$.

As it is formulated here, this result is an easy corollary from Theorem 1.1 in Krengel (1985), p. 196, where it is formulated in more general setting - for some operators (contractions), acting on finite measure space. Reduction to probability space and shift operators is standard, and that the limit is zero (in Krengel (1985) the existence of a limit is stated) follows from ergodicity of the random field under consideration.

Having these two results on SLLN for linear random fields it is clear that by using BND we can not get such general and strong results. Thus, application of BND to prove SLLN for linear random fields (the same remark can be applied for linear processes, too) can be justified only from methodological point of view: if one wants (for some reasons) to avoid ergodic theory, one can reduce the problem of SLLN for linear random fields to the case of SLLN for multi-indexed sums of i.i.d. random variables and use (comparatively elementary) Smythe (1973) result and elementary moment inequalities to estimate remainder term appearing in BND (as it was done in Paulauskas (2009)). We formulate one more such result in the case $d = 2$ in the next proposition. In all aspects this result is weaker comparing with theorems formulated above, only the moment condition is only very little stronger than (11), but it is necessary to note that to achieve such condition instead of the moment inequality we used a result from ergodic theory. The obtained result is stronger than results in Paulauskas (2009), and, in a sense, it demonstrates the limits of the BND approach in the problem. Now we return to a linear random field defined in (1) and we denote

$$S_{\mathbf{n}}^{(1)} = \sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}}. \tag{12}$$

Proposition 3. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^2$ be i.i.d. random variables with $E\varepsilon_{\mathbf{0}} = 0$. Suppose that for some $1 < p \leqslant 2$ moment condition $E|\varepsilon_{\mathbf{0}}|^p < \infty$ and condition $\mathcal{L}_p$ (with summation only over positive quadrant) holds. Then, under* (10)*, SLLN for $S_{\mathbf{n}}^{(1)}$ holds.*

2 Proofs

Before the proof of theorems we shall formulate BND, as it was stated in Paulauskas (2009). Let $\mathbf{L} = (L_1, L_2)$ be lag operators, defined by formulas:

$$L_1 \varepsilon_{\mathbf{t}} = \varepsilon_{t_1 - 1, t_2}, \; L_2 \varepsilon_{\mathbf{t}} = \varepsilon_{t_1, t_2 - 1}.$$

Let us set

$$\Phi(\mathbf{L}) = \sum_{\mathbf{k} \geqslant \mathbf{0}} \varphi_{\mathbf{k}} L_1^{k_1} L_2^{k_2}.$$

To formulate BND we need more notation. Denote

$$\mu_1 = \Phi(1,1) = \sum_{\mathbf{k} \geqslant \mathbf{0}} \varphi_{\mathbf{k}},$$

$$A_2(\mathbf{L}) = \Phi^*(\mathbf{L}) \Delta_2(\mathbf{L}), \; \Delta_2(\mathbf{L}) = (1 - L_1)(1 - L_2),$$

$$\Phi^*(\mathbf{L}) = \sum_{\mathbf{k} \geqslant 0} \varphi^*_{\mathbf{k}} L_1^{k_1} L_2^{k_2}, \quad \varphi^*_{\mathbf{k}} = \sum_{\mathbf{l} \geqslant \mathbf{k} + \mathbf{1}} \varphi_{\mathbf{l}},$$

$$A_1(\mathbf{L}) = B(L_1)\Delta_1(L_1) + D(L_2)\Delta_1(L_2), \quad \Delta_1(L_i) = (1 - L_i),$$

$$B(L_1) = \sum_{l \geqslant 0} b_l L_1^l, \quad b_l = \varphi^*_{l,-1} = \sum_{k_1 \geqslant l+1, k_2 \geqslant 0} \varphi_{k_1, k_2},$$

$$D(L_2) = \sum_{l \geqslant 0} d_l L_2^l, \quad d_l = \varphi^*_{-1,l} = \sum_{k_1 \geqslant 0, k_2 \geqslant l+1} \varphi_{k_1,k_2}.$$

Now we can formulate BND as it was stated in Paulauskas (2009).

Proposition 4. *The following identity holds*

$$\Phi(\mathbf{L}) = \mu_1 + A_2(\mathbf{L}) - A_1(\mathbf{L}). \tag{13}$$

The relations

$$\sum_{\mathbf{k} \geqslant \mathbf{0}} |\varphi^*_{\mathbf{k}}|^p < \infty, \ \sum_{l \geqslant 0} |b_l|^p < \infty, \ \sum_{l \geqslant 0} |d_l|^p < \infty, \ \mu_1 < \infty \tag{14}$$

hold if either condition $\mathcal{L}_p$ in case $1 \leqslant p < \infty$ or condition $\mathcal{L}_{1,p}$ in case $p < 1$ is satisfied.

Also we shall need the following simple result.

Lemma 5. *Let $Y_{\mathbf{t}}$ be a random field defined in (6), $d \geq 2$, a random field $\varepsilon_{\mathbf{t}}$ satisfies conditions of Theorem 1 and the condition $\mathcal{L}_{0,1}$ holds. Then for all $\mathbf{t} \in \mathbb{Z}^d$*

$$E|Y_{\mathbf{t}}|(\ln(1 + |Y_{\mathbf{t}}|))^{d-1} < \infty. \tag{15}$$

Proof of Lemma 5. Let us denote $g(x) = |x|(\ln(1+|x|))^{d-1}$. This function is convex and increasing for $x \geq 0$, therefore using obvious inequality $(1 + cx) \leq (1 + x)(1 + c)$ we have for all $c > 0, \ x > 0$

$$g(cx) \leq 2^{d-2}(cg(x) + xg(c)).$$

We must prove the boundedness of $Eg(Y_{\mathbf{t}})$. Let us denote $c = \sum_{\mathbf{k} \in \mathbb{Z}^d} |\varphi_{\mathbf{k}}|$, then, using the fact that g is convex and increasing, we have

$$\begin{aligned}
Eg(Y_{\mathbf{t}}) &= Eg(|\sum_{\mathbf{k} \in \mathbb{Z}^d} \varphi_{\mathbf{k}} \varepsilon_{\mathbf{t}-\mathbf{k}}|) \leqslant Eg(|\sum_{\mathbf{k} \in \mathbb{Z}^d} |\varphi_{\mathbf{k}}||\varepsilon_{\mathbf{t}-\mathbf{k}}|) \\
&= Eg(c \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{|\varphi_{\mathbf{k}}|}{c} |\varepsilon_{\mathbf{t}-\mathbf{k}}|) \\
&\leqslant 2^{d-2} \Big\{ cEg(\sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{|\varphi_{\mathbf{k}}|}{c} |\varepsilon_{\mathbf{t}-\mathbf{k}}|) + g(c) E \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{|\varphi_{\mathbf{k}}|}{c} |\varepsilon_{\mathbf{t}-\mathbf{k}}| \Big\} \\
&\leqslant 2^{d-2} \Big\{ cE \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{|\varphi_{\mathbf{k}}|}{c} g(|\varepsilon_{\mathbf{t}-\mathbf{k}}|) + g(c) \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{|\varphi_{\mathbf{k}}|}{c} E|\varepsilon_{\mathbf{t}-\mathbf{k}}| \Big\} \\
&\leqslant 2^{d-2} \Big\{ cEg(|\varepsilon_{\mathbf{0}}|) + g(c) E|\varepsilon_{\mathbf{0}}| \Big\} < \infty.
\end{aligned}$$

The lemma is proved.

Proof of Theorem 1. Theorem 1 directly follows from Theorem A. Namely, the random field $\{Y_{\mathbf{t}}\}$ is strictly stationary and ergodic (as a function of a random field $\{\varepsilon_{\mathbf{t}}\}$), $EY_{\mathbf{0}} = 0$, from Lemma 5 it follows that $Y_{\mathbf{0}} \in L \log L^{d-1}$. Hence we can apply Theorem A.

Proof of Theorem 2. Let, for simplicity, agree to write $[\mathbf{0}, \mathbf{n}]^c$ for a set of those $\mathbf{k} \in (\mathbb{Z}^+)^d$, which satisfy $\mathbf{k} \not\leq \mathbf{n}$. It is easy to see that the relation

$$T_{\mathbf{n}} := |\mathbf{n}|^{-1} S_{\mathbf{n}} \overset{a.s.}{\to} 0,$$

when $|\mathbf{n}| \to \infty$, is equivalent to the following condition: for each $\epsilon > 0$ there exists $\mathbf{n}^{(\epsilon)} = (n_1^{(\epsilon)}, \ldots, n_d^{(\epsilon)})$ such that

$$P\{|T_{\mathbf{n}}| < \epsilon, \forall \mathbf{n} \in [\mathbf{0}, \mathbf{n}^{(\epsilon)}]^c\} = 1.$$

At first we prove the theorem in the case $d = 2$. From Theorem 1 it follows that

$$T_{\mathbf{n}} \overset{a.s.}{\to} 0,$$

as $\mathbf{n} \to \infty$, that is, $n_i \to \infty,\ i = 1, 2$. Let $\Omega_0,\ P(\Omega_0) = 1$, be a set of those ω, for which this relation holds. (Here $(\Omega, \mathcal{F}, P)$ is a probability space on which all random variables under consideration are defined.) Let us fix $\epsilon > 0$. Then $\forall\ \omega \in \Omega_0$, there exists $\mathbf{n}^{(\epsilon)} = (n_1^{(\epsilon)}, n_2^{(\epsilon)})$ such that, $\forall\ \mathbf{n} \geq \mathbf{n}^{(\epsilon)}$, $|T_{\mathbf{n}}| < \epsilon$. Let us take some $1 \leq m \leq n_2^{(\epsilon)} - 1$ and consider sums

$$T_{(n_1,m)} = \frac{1}{n_1 m} \sum_{\mathbf{k} \leq (n_1,m)} Y_{\mathbf{k}} = \frac{1}{m} \sum_{j=1}^{m} \Big(\frac{1}{n_1} \sum_{k=1}^{n_1} Y_{(k,j)} \Big).$$

The random field $\{Y_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}^2\}$, as a function (under very mild conditions on a function; in our case it is linear function) of a mixing field $\{\varepsilon_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2\}$, is mixing, too. Therefore for any fixed j a random process $\{Y_{(k,j)}, k \in \mathbb{Z}\}$ is also mixing, hence it is ergodic, and we get

$$T_{(n_1,m)} \overset{a.s.}{\to} 0, \quad \text{as } n_1 \to \infty.$$

Let $\Omega_{1,m},\ P(\Omega_{1,m}) = 1$ be a set of those ω for which this relation holds. Taking the same $\epsilon > 0$ we can find $n_{\epsilon,m}$ such that for $n_1 \geq n_{\epsilon,m}$ and all $\omega \in \Omega_{(1,m)}$, we have $|T_{(n_1,m)}| < \epsilon$. Now for some $1 \leq r \leq n_1^{(\epsilon)} - 1$ in a similar way we consider sums $T_{(r,n_2)}$, and (for the same ϵ) we introduce a set $\Omega_{2,r},\ P(\Omega_{2,r}) = 1$, and number $n_{r,\epsilon}$ such that, for all $\omega \in \Omega_{(2,r)}$, and all $n_2 \geq n_{r,\epsilon}$, we have $|T_{(r,n_2)}| < \epsilon$. Let us take $\bar{\mathbf{n}}^{(\epsilon)} = (\bar{n}_1^{(\epsilon)}, \bar{n}_2^{(\epsilon)})$ with

$$\bar{n}_1^{(\epsilon)} = \max\{n_1^{(\epsilon)}, n_{\epsilon,m} : m = 1, \ldots, n_2^{(\epsilon)} - 1\},$$

$$\bar{n}_2^{(\epsilon)} = \max\{n_2^{(\epsilon)}, n_{r,\epsilon}, r = 1, \ldots, n_1^{(\epsilon)} - 1\}.$$

Let us denote

$$\bar{\Omega} = \Omega_0 \cap \big(\cap_{m=1}^{n_2^{(\epsilon)}-1} \Omega_{1,m} \big) \cap \big(\cap_{r=1}^{n_1^{(\epsilon)}-1} \Omega_{2,r} \big).$$

It is clear that $P(\bar{\Omega}) = 1$, and from construction it follows that for all $\omega \in \bar{\Omega}$ and for all $\mathbf{n} \in [\mathbf{0}, \bar{\mathbf{n}}^{(\epsilon)}]^c$,

$$|T_{\mathbf{n}}| < \epsilon.$$

Thus, we have proved the theorem in the case $d = 2$.

Now, using mathematical induction, we shall prove the general case $d > 2$. Let us assume that the statement of the theorem is true for all dimensions $k \leq d-1,\ d \geq 3$. We prove then that the statement of the theorem holds for dimension d. The proof is similar to that in the case $d = 2$, only notations are more complicated. Let us denote $\mathbf{n}(k) = (n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_d)$ and $(\mathbf{n}(k), m) = (n_1, \ldots, n_{k-1}, m, n_{k+1}, \ldots, n_d),\quad k = 1, \ldots, d$. Again, by using theorem 1, we have that SLLN is valid for $S_{\mathbf{n}}$ as $\mathbf{n} \to \infty$. Let $\Omega_0,\ P(\Omega_0) = 1$, be a set of those ω, for which this relation holds. Let us fix $\epsilon > 0$. Then, $\forall\ \omega \in \Omega_0$, there exists $\mathbf{n}^{(\epsilon)} = (n_1^{(\epsilon)}, \ldots, n_d^{(\epsilon)})$ such that, $\forall\ \mathbf{n} \geq \mathbf{n}^{(\epsilon)}$,

$$|T_{\mathbf{n}}| < \epsilon.$$

Fix some $1 \leq k \leq d$ and take some $1 \leq m \leq n_k^{(\epsilon)} - 1$. Consider sums

$$\begin{aligned} T_{(\mathbf{n}(k),m)} &= \frac{1}{|\mathbf{n}(k)|m} \sum_{\mathbf{r} \leqslant (\mathbf{n}(k),m)} Y_{\mathbf{r}} \\ &= \frac{1}{m} \sum_{j=1}^{m} \frac{1}{|\mathbf{n}(k)|} \sum_{\mathbf{r}(k) \leqslant \mathbf{n}(k)} Y_{(\mathbf{r}(k),j)}, \end{aligned}$$

where $|\mathbf{n}(k)| = |\mathbf{n}|/n_k$. Then, by using the same argument as in the proof of the case $d = 2$, we get that $\{Y_{(\mathbf{n}(k),j)}, \mathbf{n}(k) \in \mathbb{Z}^{d-1}\}$, for a fixed j, is also mixing. Therefore, from the induction assumption for a fixed m we get,

$$T_{(\mathbf{n}(k),m)} \stackrel{a.s.}{\to} 0, \quad \text{as } |\mathbf{n}(k)| \to \infty.$$

Denote by $\Omega_{(k,m)}$, $P(\Omega_{(k,m)}) = 1$, the set where this convergence for $T_{(\mathbf{n}(k),m)}$ holds. Taking the same $\epsilon > 0$, which we fixed at the beginning of the proof, we will find $\mathbf{n}(k)^{(\epsilon,m)} = (n_1^{(\epsilon,m)}, \dots, n_{k-1}^{(\epsilon,m)}, n_{k+1}^{(\epsilon,m)}, \dots, n_d^{(\epsilon,m)})$ such that for all $\mathbf{n}(k) \in [\mathbf{0}, \mathbf{n}(k)^{(\epsilon,m)}]^c$ and for all $\omega \in \Omega_{(k,m)}$, $|T_{(\mathbf{n}(k),m)}| < \epsilon$. Then by applying the same argument to the other coordinates we will get the set of pairs:

$$\{(\Omega_{(1,m_1)}, \mathbf{n}(1)^{(\epsilon,m_1)}) \dots, (\Omega_{(d,m_d)}, \mathbf{n}(d)^{(\epsilon,m_d)}), \quad 1 \le m_k \le n_k^{(\epsilon)} - 1\}.$$

To finish the proof we must find $\bar{\Omega}$, for which $P(\bar{\Omega}) = 1$, and $\bar{\mathbf{n}}^\epsilon$ such that, for all $\mathbf{n} \in [0, \bar{\mathbf{n}}^\epsilon]^c$ and $\omega \in \bar{\Omega}$, $|T_{\mathbf{n}}| < \epsilon$. This can be easily done, namely

$$\bar{\Omega} = \Omega_{\mathbf{0}} \cap_{k=1}^{d} \big(\cap_{m_k=1}^{n_k^\epsilon - 1} \Omega_{(k,m_k)} \big).$$

and

$$\bar{\mathbf{n}}^\epsilon = (\bar{n}_1^\epsilon, \dots, \bar{n}_d^\epsilon), \quad \bar{n}_k^\epsilon = \max \big(n_k^\epsilon, \max_{j \ne k} n_{j,k}^\epsilon \big),$$

$$n_{j,k}^\epsilon = \max_{m \le n_j^{(\epsilon)} - 1} n_j^{(\epsilon,m)}.$$

The theorem is proved.

Proof of Proposition 3. Applying (13) to the sum in (27) we get

$$S_{\mathbf{n}}^{(1)} = \mu_1 Z_{\mathbf{n}} + R_{\mathbf{n}}, \quad Z_{\mathbf{n}} = \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}}, \tag{16}$$

where

$$\begin{aligned} R_{\mathbf{n}} &= \xi_{n_1,n_2} - \xi_{n_1,0} - \xi_{0,n_2} + \xi_{0,0} \\ &+ \eta_{n_1,n_2} - \eta_{0,n_2} + \zeta_{n_1,n_2} - \zeta_{n_1,0}, \end{aligned} \tag{17}$$

$$\xi_{\mathbf{t}} = \Phi^*(\mathbf{L})\varepsilon_{\mathbf{t}} = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}}^* \varepsilon_{\mathbf{t}-\mathbf{k}},$$

$$\eta_{t_1,n_2} = \sum_{t_1=1}^{n_2} \bar{\varepsilon}_{t_1,t_2}, \quad \bar{\varepsilon}_{t_1,t_2} = B(L_1)\varepsilon_{t_1,t_2} = \sum_{l \geqslant 0} b_l \varepsilon_{t_1-l,t_2},$$

$$\zeta_{n_1,t_2} = \sum_{t_1=1}^{n_1} \hat{\varepsilon}_{t_1,t_2}, \quad \hat{\varepsilon}_{t_1,t_2} = D(L_2)\varepsilon_{t_1,t_2} = \sum_{l \geqslant 0} d_l \varepsilon_{t_1,t_2-l}.$$

$Z_{\mathbf{n}}$ is a sum of i.i.d. mean zero random variables $\varepsilon_{\mathbf{t}}$ with $E|\varepsilon_{\mathbf{0}}|^p < \infty$, for some $1 < p \leqslant 2$, therefore SLLN for $Z_{\mathbf{n}}$ holds. Then, in order to prove the proposition, we must prove that

$$|\mathbf{n}|^{-1} R_{\mathbf{n}} \xrightarrow{a.s.} 0, \quad \text{as } |\mathbf{n}| \to \infty. \tag{18}$$

We have

$$R_{\mathbf{n}} = R_{\mathbf{n}}^1 + R_{\mathbf{n}}^2 + R_{\mathbf{n}}^3, \tag{19}$$

where $R_{\mathbf{n}}^1 = \xi_{n_1,n_2} - \xi_{n_1,0} - \xi_{0,n_2} + \xi_{0,0}$, $R_{\mathbf{n}}^2 = \eta_{n_1,n_2} - \eta_{0,n_2}$, $R_{\mathbf{n}}^3 = \zeta_{n_1,n_2} - \zeta_{n_1,0}$, therefore we must prove the relation (18) for each $R_{\mathbf{n}}^i, \ i = 1, 2, 3$. We start with $R_{\mathbf{n}}^1$ and prove that

$$\sum_{\mathbf{n} \geqslant \mathbf{1}} P(||\mathbf{n}|^{-1} \xi_{\mathbf{n}}| > \epsilon) < \infty. \tag{20}$$

Since $\varepsilon_{\mathbf{t}}$ are i.i.d. and $E\varepsilon_{\mathbf{t}} = 0$, the moment inequality for $\xi_{\mathbf{t}} = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}}^* \varepsilon_{\mathbf{t}-\mathbf{k}}$ yields

$$E|\xi_{\mathbf{t}}|^p \leqslant C \sum_{\mathbf{k} \geqslant 0} |\varphi_{\mathbf{k}}^*|^p E|\varepsilon_{\mathbf{t}-\mathbf{k}}|^p \leqslant C \sum_{\mathbf{k} \geqslant 0} |\varphi_{\mathbf{k}}^*|^p < \infty. \tag{21}$$

Therefore,

$$\sum_{\mathbf{n} \geqslant \mathbf{1}} P(||\mathbf{n}|^{-1} \xi_{\mathbf{n}}| > \epsilon) \leqslant E|\xi_{\mathbf{1}}|^p \sum_{\mathbf{n} \geqslant \mathbf{1}} \epsilon^{-p} |\mathbf{n}|^{-p} < \infty, \tag{22}$$

thus we get

$$|\mathbf{n}|^{-1} \xi_{\mathbf{n}} \xrightarrow{a.s.} 0. \tag{23}$$

Clearly, the same relation holds for $\xi_{n_1,0}$, ξ_{0,n_2}, $\xi_{0,0}$, and we get that

$$R_{\mathbf{n}}^1 \xrightarrow{a.s.} 0. \tag{24}$$

Unfortunately, the moment inequality is too rough for other two terms $R_{\mathbf{n}}^2$ and $R_{\mathbf{n}}^3$ (this was noted in Paulauskas (2009)), since, using low order moments for $\zeta_{\mathbf{t}}$ and $\eta_{\mathbf{t}}$, we get divergent series in (20), or we must require existence of moment of the order $2 + \delta, \ \delta > 0$. Therefore we shall use one result from ergodic theory. Let us consider random variable $\eta_{\mathbf{t}}$ and let us denote

$$K_{\mathbf{n}} = \frac{1}{|\mathbf{n}|} \eta_{\mathbf{n}} = \frac{1}{n_1 n_2} \sum_{t_2=1}^{n_2} \bar{\varepsilon}_{n_1,t_2}. \tag{25}$$

Taking into account the definition of $\bar{\varepsilon}_{\mathbf{t}}$, we have

$$\begin{aligned} K_{\mathbf{n}} &= \frac{1}{n_1 n_2}\sum_{t_2=1}^{n_2}\Big(\sum_{l\geqslant 0} b_l \varepsilon_{n_1-l,t_2}\Big) = \frac{1}{n_1 n_2}\sum_{t_2=1}^{n_2}\Big(\sum_{k=-\infty}^{n_1} b_{n_1-k}\varepsilon_{k,t_2}\Big) \\ &= \frac{1}{n_1}\sum_{k=-\infty}^{n_1} b_{n_1-k}\Big(\frac{1}{n_2}\sum_{t_2=1}^{n_2}\varepsilon_{k,t_2}\Big). \end{aligned} \tag{26}$$

Therefore, denoting $f_k = \sup_{n_2}|n_2^{-1}\sum_{t_2=1}^{n_2}\varepsilon_{k,t_2}|$, we get

$$|K_{\mathbf{n}}| \leqslant \frac{1}{n_1}\sum_{k=-\infty}^{n_1}|b_{n_1-k}|\sup_{n_2}\Big|\frac{1}{n_2}\sum_{t_2=1}^{n_2}\varepsilon_{k,t_2}\Big| = \frac{1}{n_1}\sum_{k=-\infty}^{n_1}|b_{n_1-k}|f_k. \tag{27}$$

Applying Proposition 50.2 from Parthasarathy (1983) we get $E|f_k| < \infty$. Here it is worth to note that the same conclusion we can get under weaker moment condition $\varepsilon_{\mathbf{0}} \in L\log L$ using Exercise 50.4 from the same book, but this moment condition is insufficient in order to get (23) by means of the moment inequality. Note that (f_k) is a sequence of i.i.d. random variables, since for different k sequences $(\varepsilon_{k,t_2},\ t_2\in\mathbb{Z})$ are independent. Let us denote $X_{n_1} = \sum_{k=-\infty}^{n_1}|b_{n_1-k}|f_k$. From lemma 11 in Paulauskas (2009) we know that $\sum_{l\geqslant 0}|b_l| < \infty$, therefore

$$E|X_l| \leqslant \sum_{k=-\infty}^{l}|b_{l-k}|E|f_k| \leqslant C\sum_{k=-\infty}^{l}|b_{l-k}| < \infty. \tag{28}$$

Sequence $\{X_n,\ n\geq 1\}$ is a stationary, ergodic (since it is generated by a sequence of i.i.d. random variables) and with finite mean, thus we can write

$$\frac{1}{n_1}X_{n_1} = \frac{1}{n_1}\Big(\sum_1^{n_1}X_l - \sum_1^{n_1-1}X_l\Big) = \frac{1}{n_1}\sum_1^{n_1}X_l - \frac{1}{n_1-1}\sum_1^{n_1-1}X_l\frac{n_1-1}{n_1} \overset{a.s.}{\to} 0, \tag{29}$$

when $n_1\to\infty$. From (27) we get that, for all n_2,

$$K_{\mathbf{n}} \overset{a.s.}{\to} 0, \quad \text{as } n_1\to\infty. \tag{30}$$

Since we need to show that $K_{\mathbf{n}} \overset{a.s.}{\to} 0$, as $|\mathbf{n}|\to\infty$, it remains to consider the case, where $n_2\to\infty$ and n_1 stays bounded. If $|\mathbf{n}|\to\infty$ in a such way that n_1 is fixed and $n_2\to\infty$, then

$$\frac{1}{n_2}\sum_{t_2=1}^{n_2}\bar{\varepsilon}_{n_1,t_2} \overset{a.s.}{\to} 0.$$

If a sequence $\mathbf{n}_k = (n_{1,k}, n_{2,k}),\ k \geq 1$ is such that $1 \leq n_{1,k} \leq k_0$, k_0 is some fixed number and $n_{2,k} \to \infty$, then the sequence $K_{\mathbf{n}_k}$ can be divided into k_0 subsequences in a such way, that in each subsequence the first index of summands is fixed. Then we apply the argument used above and get the convergence to zero a.s. of each subsequence, therefore the sequence itself also converges to zero a.s. Thus we have proved (30), if $|\mathbf{n}| \to \infty$.

It is easy to see that the convergence to zero a.s. of $(n_1 n_2)^{-1}\eta_{0,n_2}$ can be proved in the same way, therefore we have

$$R_{\mathbf{n}}^2 \overset{a.s.}{\to} 0. \tag{31}$$

Due to symmetry the same consideration could be applied to $\zeta_{\mathbf{t}}$ and we get

$$R_{\mathbf{n}}^3 \overset{a.s.}{\to} 0. \tag{32}$$

Collecting (19), (24), (31), (32) we get (18), and the proposition is proved.

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References

Georgii, H.-O., 1988. Gibbs measures and phase transitions. de Gruyter, Berlin.

Krengel, U., 1985. Ergodic Theorems. de Gruyter, Berlin.

Marinucci, D., Poghosyan, S., 2001. Asymptotics for linear random fields. Statist. Probab. Lett. 51, 131–141.

Parthasarathy, K., 1983. Introduction to probability and measure. Mir, Moscow.

Paulauskas, V., 2009. On Beveridge-Nelson decomposition and limit theorems for linear random fields. J. Multivariate Analysis. doi: 10.1016/j.jmva.2009.10.001.

Phillips, P., Solo, V., 1992. Asymptotics for linear processes. Ann. Stat. 20, 971–1001.

Rieders, E., 1993. The size of the average of strongly mixing random variables. Statist. Probab. Lett. 18, 57–64.

Smythe, R., 1973. Strong law of large numbers for r-dimensional arrays of random variables. Ann. Probab. 1, 164–170.

Walters, P., 1982. An Introduction to Ergodic Theory. Springer, New York.